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XV. *On the Flow of Electricity in a uniform plane conducting Surface.*—Part I.

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[With Two Plates.]

1. The objects of the following paper are:—first, to show that the most important laws relating to the flow of electricity in a plane conducting sheet of uniform conductivity can be established by mathematical considerations of such a simple kind that there is no reason why they should not be introduced into ordinary teaching, as well as into the more complete elementary text-books of electricity; and in the second place, to describe methods that may conveniently be employed for testing experimentally the theoretical conclusions, and to record some results obtained by these methods.

The general subject treated of in this paper has attracted the attention of a considerable number of mathematicians and physicists. The earliest published investigation relating to it is contained in a remarkable memoir by Kirchhoff, which appeared in Poggendorff's *Annalen* in 1845 (vol. lxiv. p. 497). In this paper Kirchhoff established the general mathematical theory of the flow of electricity in an unlimited uniformly conducting-

sheet, and in a limited sheet with a circular boundary, with so much completeness as to leave little for others to do beyond working out the application to special cases of the general principles he laid down, or finding other methods of establishing the conclusions he deduced from them.

We cannot better indicate the general plan of Kirchhoff's investigation than by quoting the following account of it from a paper by Professor W. Robertson Smith*, to which we shall have to make further reference immediately:—"By an application of Ohm's law, he [Kirchhoff] expressed analytically the condition to be satisfied by v [the potential]. When the electricity enters and issues by a number of individual points, he found (apparently by trial) that an integral of the form $\Sigma(\alpha \log r)$, where r_1, r_2, &c. are the distances of the point (x, y) from the successive points of entrance and issue, satisfied the conditions when the plate is infinite. For a finite plate, it is necessary that the boundary of the plate should be orthogonal to the curves

$$\Sigma(\alpha \log r) = \text{const.} \quad . \quad . \quad (3)$$

He was thus led to form the orthogonal curves whose equation he gives in the form

$$\Sigma(\alpha[r, \mathrm{R}]) = \text{const.}, \quad . \quad . \quad (4)$$

where $[r, \mathrm{R}]$ is the angle between r and a fixed line R. These equations he applies to the case of a circular plate, completely determining the curves when there is one exit and one entrance point in the circumference, and showing that in any case a proper number of subsidiary points would make the equipotential lines determined by (3) cut the circumference at right angles. Kirchhoff's paper is throughout properly busied with the function v, and the stream-lines are only dealt with incidentally. There is no attempt to give a physical meaning to the equation (4)." To this we have only to add that Kirchhoff proved the accuracy of his theoretical deductions by determining experimentally the form and distribution of the equipotential lines on a circular disk with two electrodes on the edge, as well as (Pogg. *Ann.* vol. lxvii. p. 344, 1846) the strength of the current at various parts of the disk; and that, from the expression for the difference of potential between the electrodes, he deduced by Ohm's law the resistance of a circular disk with two small electrodes

* Proc. Edinb. Roy. Soc. 1869–70, pp. 79–99.

anywhere upon it. In order to test experimentally the value thus obtained, he seems to have devised independently the arrangement now commonly known as Wheatstone's bridge; but, owing to the smallness of the resistance to be measured, he was unable to obtain satisfactory results.

Soon after the publication of Kirchhoff's paper, Smaasen* gave an investigation of the flow of electricity in a plane conducting sheet, in which he takes account, in determining the potential, of the electricity given off to the air, and deduces the resistance of an infinite sheet with two small circular electrodes by a process which, though longer than that employed by Kirchhoff, may be regarded as more direct. It consists in finding the resistance of the space between two lines of flow at an infinitely small distance apart, and then extending by integration the expression thus obtained so as to make it apply to the unlimited sheet. In a subsequent paper† Smaasen determined by an analogous process the resistance of a conducting sphere, or of an unlimited conductor of three dimensions‡. Smaasen's treatment of the subject is, like Kirchhoff's, based chiefly upon the mode of distribution of the potential; the only investigation we are acquainted with which deals specially with lines of flow is contained in the paper by Professor W. Robertson Smith from which we have already quoted. The starting-point adopted by Professor Smith is the same as that from which we have set out in the following communication; and, indeed, we found, after making some progress in our own work, that several of our demonstrations (which we at first thought were new) had been already given by him, while all the chief conclusions were, as we have said above, implicitly contained in Kirchhoff's original memoir. Consequently, although the present paper contains a few minor results which, so far as we know, have not been pointed out ex-

* Pogg. *Ann.* vol. lxix. p. 161.

† Pogg. *Ann.* vol. lxxii. p. 435.

‡ About the same time, the same subject was taken up by Ridolfi of Florence (*Il Cimento*, An. V. 1847, May–June), whose paper, however, we only know from the references of Beetz (in Dove's *Repertorium*, vol. viii. p. 147) and Poggendorff (Pogg. *Ann.* vol. lxxii. p. 449). No reference to this paper is given either in the Royal Society's Catalogue of Scientific Papers, or in the carefully compiled "Bibliographie" of the mathematical theory of the voltaic pile in Verdet's *Conférences de Physique* (*Œuvres de* Verdet, vol. iv. p. 351).

plicitly before, we do not claim for its contents any essential novelty; and our only reason for venturing to publish it is that the mode of treatment adopted in it has enabled us to establish by strictly elementary mathematical methods, and so to render accessible to a greater number of readers, many of the conclusions which previous writers had arrived at by more abstruse processes.

2. In order to avoid ambiguity, it may be well to begin by giving a few general explanations and definitions of terms that will be frequently used in what follows. It will be assumed that, in all cases considered, the flow of electricity has attained a *stationary* condition—that is to say, that on the whole there is neither gain nor loss of electricity at any part of the sheet, but that at every instant the quantity that enters any part and the quantity that leaves it are equal. Positive electrodes, or such as supply positive electricity to the sheet, will be spoken of as "sources," while negative electrodes will be termed "sinks." The rate at which a source or a sink supplies or removes electricity will be spoken of as the "strength" of the source or sink.

Through every point of the sheet a straight line can be drawn such that no electricity crosses it at that point; the direction of a line so drawn through a given point is the "direction of the current" at that point. A "line of flow" or "stream-line" is a line drawn so that no electricity crosses it anywhere; or, in other words, it is a line whose direction at all points coincides with the direction of the current. The quantity of electricity flowing between two consecutive lines of flow is everywhere the same; for since no electricity crosses either of them, whatever quantity starts between them must remain between them throughout their whole course. It may be useful to observe that flow-lines are not loci of constant strength of flow, they are lines of direction simply.

The "strength of the current" at any point is given by the quantity of electricity which crosses, in unit time, a short line normal to the flow-line through the point, divided by the length of the line.

If a line be drawn so as to intersect the lines of flow everywhere at right angles, there will be no flow of electricity along it; but since the sheet on which the line is drawn is a conductor, this implies that all points of the line are at the same potential, or that the line is an "equipotential line."

3. It is obvious that a line of flow and an equipotential line might be drawn through every point of a conducting sheet traversed by a current, so that, if it were possible to draw them all, the sheet would be completely covered by two sets of lines cutting each other at right angles. The general distribution, however, of the lines of each set can be correctly shown without drawing more than a limited number, if those that are drawn are selected according to a definite and easily recognizable rule. For this purpose the most convenient rule to adopt in relation to the lines of flow, is to place them so that the total flow of electricity between each pair of consecutive lines is the same—and with regard to the equipotential lines, to place them so that the change of potential in passing from any one to the next consecutive line is the same. In the sequel, whenever a *system* of lines of flow, or of equipotential lines, is referred to, it is to be understood that the lines are placed so as to fulfil the above conditions.

One Pole in an Infinite Sheet.

4. In this case it is evident that the lines of flow are straight lines radiating out from the pole, and that the spaces between each pair of consecutive lines will convey equal currents if each line makes the same angle with the next. If a circle of radius r be drawn about the pole as centre, the quantity of electricity which crosses the whole circumference in unit of time, will evidently be the same whatever the value of r, and will be equal to the quantity Q supplied in the same time by the source. Hence, at distance r from the source, the strength of the current will be

$$\frac{Q}{2\pi r},$$

or inversely proportional to the distance r.

It is evident also, either from general considerations of symmetry, or from the condition that equipotential lines and lines of flow must intersect each other at right angles, that the equipotential curves for the case we are considering are circles having the source at their common centre.

5. If R be the resistance to the radial flow of electricity across the annular portion of the sheet bounded by circles at the potentials V_1 and V_2, we have, by Ohm's law,

$$V_1 - V_2 = QR;$$

whence, since Q is constant, equal differences of potential will be found at the boundaries of annular belts of equal resistance. The relation between the radii of successive equipotential circles with a constant difference of potential may be deduced from this condition as follows:—The resistance of any very narrow belt must be proportional to its breadth, or the difference between the radii of its external and internal circumferences; and it must be inversely proportional to the mean circumference, which, seeing that the circumference varies directly as the radius, will be simply the arithmetic mean of the external and internal circumferences. That is, if r_1, r_2, r_3 are radii of successive circles, these circles will bound spaces of equal resistance if

$$\frac{r_2-r_1}{\pi(r_2+r_1)}=\frac{r_3-r_2}{\pi(r_3+r_2)},$$

or

$$r_1r_3=r_2^2.$$

Hence the radii of successive equipotential circles form a geometrical progression whose common ratio may be called μ.

6. To find the resistance of a belt bounded by concentric circles whose radii are r_1 and r_2 respectively, let

$$\frac{r_2}{r_1}=\mu^n,$$

so that there are n equipotential circles contained in the belt, where

$$n=\frac{1}{\log\mu}\log\frac{r_2}{r_1}.$$

Denote the thickness of the conducting sheet by δ and its conductivity by κ; then, by the last section, the resistance of a narrow ring whose bounding circles have the radii 1 and $1+\Delta r(=\mu)$ will be

$$R_0=\frac{1}{\kappa\delta}\cdot\frac{\Delta r}{\frac{1}{2}(2+\Delta r)2\pi}=\frac{1}{\pi\kappa\delta}\cdot\frac{\Delta r}{2+\Delta r}.$$

But the whole belt we are considering is made up of n rings, each of the resistance R_0. So the resistance of the whole belt is

$$R=nR_0=\frac{1}{(2+\Delta r)\pi\kappa\delta}\cdot\frac{\Delta r}{\log(1+\Delta r)}\log\frac{r_2}{r_1};$$

or, letting Δr decrease without limit,

$$R=\frac{1}{2\pi\kappa\delta}\log\frac{r_2}{r_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

7. Putting this value of R into the expression for the difference of potential of two circles of radii r_1 and r_2, we have

$$V_1 - V_2 = \frac{Q}{2\pi\kappa\delta} \cdot \log\left(\frac{r_2}{r_1}\right),$$

whence the potential at any point of the sheet is at once given in terms of the strength of the source and the potential at unit distance from the source. Thus let V_1 be the potential at unit distance, then the potential V at any distance r becomes

$$V = V_1 - \frac{Q}{2\pi\kappa\delta} \log r. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

8. The same result may be obtained without previously calculating the resistance of a belt of finite breadth as follows. The radii of successive equipotential circles, beginning at the pole itself, are (§ 5)

$$\mu^{-\infty} \ldots \mu^{-2}, \mu^{-1}, 1, \mu, \mu^2 \ldots \mu^{\infty}.$$

Let the common difference of potential between consecutive circles be Δv; then, if $r = \mu^n$, the potential at distance r is less than the potential at unit distance by n times Δv, or

$$V_r = V_1 - n\Delta v = V_1 - \frac{\Delta v}{\log \mu} \log r.$$

But $\frac{\Delta v}{\log \mu}$ is constant; and putting $1 + dr$ for μ, where dr is the breadth of the infinitely narrow belt whose inner boundary is the circle of unit radius, we have

$$\frac{\Delta v}{\log \mu} = \frac{dv}{\log (1 + dr)} = \frac{dv}{dr}.$$

Multiplying this by the circumference of the belt (2π) and by the conductivity and thickness of the sheet, we get the strength of the current across the whole belt, or

$$2\pi\kappa\delta \frac{dv}{dr} = Q,$$

$$\frac{\Delta v}{\log \mu} = \frac{Q}{2\pi\kappa\delta},$$

whence

$$V = V_1 - \frac{Q}{2\pi\kappa\delta} \log r,$$

as before.

Two equal Opposite Poles in an Infinite Sheet.

9. If there are two or more electrodes in an infinite uniform conducting sheet, the strength and direction of the current and the potential at any part of the sheet are obtained by the simple superposition of the effects which each electrode would produce at that part if it were the only electrode in the sheet. This might be regarded as probable *à priori*; and it is proved by experiment to be the case. Hence the effect of any number of poles in a sheet may be deduced by properly extending the conclusions already arrived at with regard to a single pole. We will first discuss the case of a single source and a single sink of equal strength.

10. *Lines of Flow.*—Let A (Plate IV. fig. 1) be the source, and B the sink. The flow-lines due to these, taken separately, would be two equiangular pencils of straight lines drawn outwards from A and inwards towards B. These two systems of lines subdivide the whole surface of the conducting sheet into quadrilaterals, such as P Q and Q R. The actual direction of the flow at any point P, due to the combined action of the source and sink, will evidently be intermediate between the directions of the primary streams A P and P B through the same point; similarly the direction of the resultant current at Q will be intermediate between the direction of the primary streams A Q and Q B; in other words, the line of resultant flow through P will enter the quadrilateral P Q at P, and the line of resultant flow through Q will pass at Q from the quadrilateral P Q into the quadrilateral Q R. And it can be shown that the points P, Q, and R are points on the same line of flow—that is, that a continuous curve can be drawn through P, Q, and R such that no electricity flows across it. Thus, let the points of intersection of the primary flow-line A P with the primary flow-lines through B which are nearest to P B on either side of it, be marked P' and $P_,$ respectively; and similarly let Q' and $Q_,$ R' and $R_,$ be the points where A Q and A R intersect the next flow-lines on either side of Q B and R B respectively. Then, since the total flow of electricity from A between the lines A P and A Q is everywhere the same and is equal to the flow towards B between the lines P B and Q B, which is likewise everywhere the same, the quantity of electricity flowing in a given time across any of the lines $P Q'$, $P_, Q$, $P P_,$, $Q' Q$, which bound the quadrilateral

P Q, is equal. Consequently, considering either of the triangles P Q′ Q or P P_1 Q, the flow inwards across P Q′ or P P_1 is equal to the flow outwards across Q′ Q or P_1 Q; and therefore there cannot on the whole be any flow across a line drawn inside the quadrilateral from P to Q. Evidently also, by drawing a sufficient number of straight lines through A in directions intermediate between A P and A Q, and an equal number of lines through B in directions lying between P B and Q B, keeping the angle between consecutive lines constant in both cases, we can subdivide the line P Q into portions as short as we please, and such that no electricity flows across any of them. Hence P and Q are points on the same line of flow; and it follows, similarly, that the point R is also situated on this line. The same kind of reasoning also proves that a second line of flow passes through the points P′, Q′, and R′, and a third through the points P_1, Q_1, and R_1. Moreover, since the strength of the flow between the lines P′ Q′ R′ and P Q R is measured by the quantity of electricity which in unit of time crosses any of the lines P′ P, P Q′, Q′ Q, Q R′, or R′ R, and the strength of the flow between the lines P Q R and P_1 Q_1 R_1 is measured by the quantity which similarly crosses any of the lines P P_1, P_1 Q, Q Q_1, Q_1 R, or R R_1, and since these quantities are equal to each other, it follows that the lines P′ Q′ R′, P Q R, and P_1 Q_1 R_1 are consecutive flow-lines of a system which divides the conducting sheet into portions each of which conveys an equal current.

It is important to observe that the reasoning employed here is general, and not limited to the special case to which it has been applied. The general conclusion to which it leads may be thus stated:—If similar * systems of lines of flow be drawn, corresponding to each of two separate systems of sources and sinks, the lines of flow which would result from the combined action of the sources and sinks of both systems will be obtained by drawing curves through the alternate angles of the quadrilaterals produced by the intersections of the two primary systems of flow-lines, in directions concurrent with both the primary flow-lines that intersect each other at each angle.

11. The method which, as we have seen, allows the flow-lines

* By *similar* systems is here to be understood systems such that the total flow between any pair of consecutive lines of the one set is the same as that between any two consecutive lines of the other set.

for two equal opposite poles to be drawn, also enables us to deduce very easily their general form. Let α be the constant angle between consecutive flow-lines of the pencil diverging from A and of that converging to B. Then, evidently,

$$\angle A P B = \angle A Q B = \angle A R B = \ldots = n\alpha,$$

where n is a constant integer. Also

$$\angle A P' B = \angle A Q' B = \ldots = (n+1)\alpha,$$

and

$$\angle A P_1 B = \angle A Q_1 B = \ldots = (n-1)\alpha.$$

Hence the lines of flow due to one source and one sink of equal strength are arcs of circles passing through the poles, each one differing from the next by a constant change ($=\alpha$) in the angle which the radii vectores from the poles make with each other; or, what comes to the same thing, they are arcs of circles cutting each other at the poles with a constant difference of angle equal to the constant difference of angle ($=\alpha$) between the rectilinear flow-lines which either the source or the sink would produce by itself.

12. The whole number of flow-lines is therefore $\frac{2\pi}{\alpha}$, or the same as the number of lines leaving the source or entering the sink when either of them is by itself in the sheet. This is evident also if we consider that infinitely near to either pole the effect of the other vanishes in comparison, and therefore the lines close to each pole are the same in all respects as they would be if the other pole were absent.

13. The circular form of the flow-lines for the case we are considering can be demonstrated in various other ways. We will give here one additional proof, on account of its great simplicity.

The flow through any point P due to a source at A and a sink at B, being the resultant of the currents through the same point due to A and B taken separately, will be represented in strength and direction by the third side of a triangle whose other two sides represent the currents from A and towards B, respectively, in strength and direction. But the strengths of the two component currents are inversely as the distances A P and B P respectively (§ 4): hence the following construction (fig. 2). From B draw B T parallel to A P, and make its length a third proportional to A P and P B; then P T gives the direction of

the flow at P, and its length is proportional to the strength of the current at P. The similarity of the triangles A P B and P B T gives the angle B P T equal to the angle P A B; and consequently (converse of Euclid, III. 32) the locus of P is a circle through A and B.

14. It was shown in § 10 how a system of lines of flow can be traced out by successive points. To be able to draw them continuously with compasses we only require to know the position of the centres; and these are easily found from the following considerations. Since the circles of which the flow-lines are arcs pass through the poles A and B, their centres lie in the straight line at right angles to A B, through O its middle point. If C (fig. 3) be the centre of the circle which gives the flow-line through any point P, the angle at C is equal to the angle at P —the angle characteristic of the given flow-line; and therefore the angle O A C is the complement of the angle at P. Putting A B$=2a$, we have

$$O C = a \,.\, \tan O A C = a \,.\, \tan\left(\frac{\pi}{2} - A P B\right).$$

Let the number of lines to be drawn be m, then the constant difference between the angles contained under consecutive lines will be

$$\frac{2\pi}{m} = \alpha,$$

and the several lines will be given by making the angle at the circumference successively equal to

$$0,\ \alpha,\ 2\alpha \ldots,\ \pi-\alpha,\ \pi,\ \pi+\alpha, \ldots 2\pi-2\alpha,\ 2\pi-\alpha,\ 2\pi,$$

or, what is the same thing, to

$$0,\ \alpha,\ 2\alpha, \ldots \pi-\alpha,\ -\pi,\ -(\pi-\alpha), \ldots -2\alpha,\ -\alpha,\ 0,$$

where the values 0, π, $-\pi$, and 2π are represented by the straight line through A and B, and negative angles indicate flow-lines passing on the lower side of A B. But, as is evident from the figure, the same circle gives the two flow-lines whose characteristic angles are $n\alpha$ and $-(\pi-n\alpha)$; hence the number of circles to be drawn or of centres to be found is equal to half the number of flow-lines; and we need only consider those characterized by angles between 0 and π, of which the complements are

$$\frac{\pi}{2},\ \frac{\pi}{2}-\alpha,\ \ldots\ \alpha,\ 0,\ -\alpha,\ \ldots\ -\left(\frac{\pi}{2}-\alpha\right),\ -\frac{\pi}{2}.$$

Consequently the required centres are obtained by drawing lines from A making the above angles with A B, and letting them intersect the perpendicular to A B through O; or, without measuring angles, by laying off from O in both directions along the perpendicular to A B lengths proportional to

$$0,\ a\tan\alpha,\ a\tan 2\alpha,\ \ldots\ a\tan\left(\frac{\pi}{2}-\alpha\right),\ a\tan\frac{\pi}{2}.$$

Plate IV. shows a system of lines of flow for which $\alpha=20°$.

15. The *strength of the current* at any point P due to a source at A and an equal sink at B is represented in the construction given in § 13 by the length of the line P T (fig. 2); but by the similarity of the triangles P B T and A P B,

$$\frac{PT}{BT}=\frac{AB}{PB};$$

or, putting $AB=2a$, $BT=\frac{1}{r}$, and $PB=r'$, the strength of the current is

$$s=\frac{Q}{2\pi}.PT=\frac{Q}{2\pi}\cdot\frac{2a}{rr'}. \quad \ldots\ldots \quad (3)$$

Hence, for given poles at a given distance apart, the strength of the current is the same at all points of the sheet for which the product of the distances from the two poles is constant; or the loci of equal flow are a system of *lemniscates**.

16. *Equipotential Lines.*—The flow-lines for the case of one source and one equal sink having been determined, the form of the equipotential lines is at once given by the consideration that the two sets of lines cut each other orthogonally (§ 2); and it is a well-known geometrical result that the system of lines orthogonal to the system of circles which, as has been seen, represents the flow-lines for this case, is another system of circles having

* The strength of the current at various parts of a circular disk with equal opposite poles on the edge was examined experimentally by Kirchhoff, by suspending a small magnet, with a mirror attached, close above the disk, and was found to agree closely with the results of calculation. Mach has since given a more direct proof that the lines passing through points of equal flow are lemniscates. A disk of silver-leaf was coated with a thin film of wax; and on allowing a strong current to traverse it for a few moments, the wax was melted within a space bounded by a lemniscate and the edge of the disk (Carl's *Repertorium f. experim. Physik*, vol. vi. p. 11, 1870).

their centres on the line through the poles, each circle cutting this line once internally and once externally in points situated harmonically with respect to its extremities. The simplest general expression for such a system of circles is the equation

$$\frac{r}{r'}=c,$$

where c is a quantity which remains constant for each circle but varies from each circle to the next, and r and r' are the distances from any point of the curve to the source and sink respectively.

These general properties of the equipotential lines are easily established by referring to the construction employed in § 13. We there saw that P T (fig. 2) is a tangent to the line of flow through P; and consequently it is normal to the equipotential line through the same point. If we produce P T to cut A B produced in C, we have the triangles B C P and P C A similar, and

$$CP^2=CA\,.\,CB;$$

whence it appears that if tangents to the lines of flow be drawn from any point C in A B produced, the distance from C to the points where these tangents touch the lines of flow is constant and depends only on the distances of the point C from A and B respectively. Therefore, if a circle be drawn with centre C and radius C P, it cuts all the lines of flow at right angles, and is consequently an equipotential line. If it is only required to find the centre of the equipotential circle through any point P, the simplest method is to make an angle B P C equal to the angle B A P; then the point where P C and A B intersect is the centre to be found.

The similarity of the triangles B C P and P C A also gives

$$\frac{AP}{PB}=\frac{r}{r'}=\frac{AC}{CP};$$

or the ratio of the *radii vectores* from the two poles is constant for a given circle, as already stated.

17. The above method suffices for drawing the equipotential lines through any number of given points, but not for placing them *systematically* (or so that the difference of potential between consecutive lines may be constant). For this purpose we may have recourse to a process of superposition of the same kind as that employed (§ 10) for placing the lines of flow.

We have seen (§ 5) that the equipotential lines for a single pole are concentric circles, and that the radii of consecutive circles form a geometrical progression. To find the system of equipotential lines for two equal opposite poles, it is only needful to draw for each pole separately a system of equipotential lines with the same difference of potential between any one line and the next as it is intended that there should be between the lines of the combined system, and then to draw lines through the intersections of the two overlapping systems of circles thus obtained, taking care, in going from one intersection to the next, that the changes of potential are in opposite directions for the two primary systems taken separately. Thus, let the lines 1, 2, 3, 4 (fig. 4) represent portions of equipotential lines due to a source at A; and $1'$, $2'$, $3'$, $4'$ portions of equipotential lines due to a sink at B; and let the potential of the line 1 be V, and let that of the line $1'$ be V'; further, let the change of potential in passing from any one line to the next in either system be v, so that the potentials of the lines 2, 3, and 4 are $V-v$, $V-2v$, and $V-3v$ respectively, and the potentials of $2'$, $3'$, and $4'$ are $V'+v$, $V'+2v$, and $V'+3v$ respectively. Then, at the points where 1 and $1'$, 2 and $2'$, 3 and $3'$, 4 and $4'$ respectively intersect each other, the potentials will be the sums of the potentials of the intersecting lines; and therefore the potential at all these points is the same, namely $V+V'$. Consequently P, Q, R, and S are points on the same equipotential line. Similarly it follows that P_1, Q_1, and R_1 are points on the line whose potential is $V+V'+v$; and Q', R', and S' points on the line whose potential is $V+V'-v$. We thus get the potential of the resultant equipotential lines differing by the constant amount v, which is the same as the difference of potential of the lines of the two primary systems.

It is evident from this that any two systems of equipotential lines whatever, which have the same constant difference of potential, can be compounded, so as to give a single resultant system, by tracing lines through alternate angles of the quadrilaterals produced by the mutual intersection of the lines of the two systems, and also that the constant difference of potential between the lines of the resultant system will be the same as that between the lines of each of the component systems.

18. Let μ be the common ratio of the radii of the equipoten-

tial circles of the two primary systems considered in § 17, and let the radius of the circle 1 be μ^n and that of the circle 1′ be μ^m (see § 8); the radii of the successive circles of the one set are then μ^{n+1}, μ^{n+2}, . . ., and of those of the other μ^{m+1}, μ^{m+2}, . . . Hence for the ratio of the radii vectores from the two poles for the points P, Q, R, S, we have

$$\frac{\mu^n}{\mu^m}, \quad \frac{\mu^{n+1}}{\mu^{m+1}}, \quad \frac{\mu^{n+2}}{\mu^{m+2}}, \quad \frac{\mu^{n+3}}{\mu^{m+3}}$$

respectively; that is, we have for all these points the common ratio

$$\mu^{n-m}.$$

Similarly, we should find for the points P_1, Q_1, and R_1 the common ratio

$$\mu^{n-m+1},$$

and for the points Q′, R′, and S′ the common ratio

$$\mu^{n-m-1}.$$

Hence, not only are the equipotential lines for two equal and opposite poles characterized by a constant ratio of the radii vectores, as already proved (§ 16), but this ratio *changes in a constant ratio* on passing from any one line of the system to the next, the ratio of change ($=\mu$) being the same as the ratio of change of radius on passing from one line of the system due to a single pole to the next.

19. The actual potential at any point of the sheet, in terms of the distances of this point from the two poles, follows directly from equation (2) in § 7. Let r be the distance of the given point from the source, and r' its distance from the sink; put V for the potential at the point due to the source alone, and V′ for that due to the sink alone. Then we have

$$V = V_1 - \frac{Q}{2\pi\kappa\delta} \cdot \log r,$$

and

$$V' = V'_1 - \frac{Q'}{2\pi\kappa\delta} \cdot \log r';$$

and since the source and sink are equal, $V'_1 = -V_1$ and $Q' = -Q$; therefore the resultant potential, or $V + V'$, is

$$U = \frac{Q}{2\pi\kappa\delta} \cdot \log \frac{r'}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This gives the potential $=0$ at all points of the straight line equidistant from the two poles, positive on the side of this line next the source, and negative on the side next the sink. Also it shows that for equal differences of potential we must have equal differences in the value of $\log \frac{r}{r'}$, which agrees with what was proved above (§ 18).

We may write (4) thus (§ 8),

$$U=\frac{Q}{2\pi\kappa\delta}\log\mu^n=n\,.\,\Delta v\,;$$

whence, regarding the source and sink for the present as mere points, the value $n=\infty$ will correspond to the former, and $n=-\infty$ to the latter, while $n=0$ denotes the straight line at an equal distance from both.

20. Let the distances from O (the middle point of A B) of the points D and D′ (fig. 2), in which the circle of potential U cuts A B, be called l and l'; then, a being as usual half the distance between the poles,

$$\frac{a+l}{a-l}=\frac{l'+a}{l'-a}=\mu^n,$$

where μ^n is the ratio of the radii vectores of the given circle. Hence

$$ll'=a^2$$

and

$$l=a\frac{\mu^n-1}{\mu^n+1},\quad l'=a\frac{\mu^n+1}{\mu^n-1}.$$

For the radius of the circle, we have evidently

$$\rho=\tfrac{1}{2}(l'-l)=\frac{2a\mu}{\mu^{2n}-1},$$

and for the distance of its centre from O,

$$d=l+\rho=l'-\rho=a\frac{\mu^{2n}+1}{\mu^{2n}-1};$$

also

$$\rho^2=\mathrm{CA}\,.\,\mathrm{CB}=(d+a)h=d^2-a^2,$$

where $h(=d-a)$ is the distance of the centre of the equipotential circle from the nearest pole.

21. In order to draw a system of equipotential circles with compasses, it is most convenient to have given the distances of

the centres from the nearest pole, and also the distances from the same point of one of the intersections with A B, the line joining the poles—that is, in fig. 2 the distance B C and either B D or B D′. Now

$$BC = h = d - a = \frac{2a}{\mu^{2n} - 1},$$

$$BD = a - l = \frac{2a}{\mu^n + 1},$$

and

$$BD = l' - a = \frac{2a}{\mu^n - 1}.$$

Any convenient value such as 2 or $\frac{3}{2}$ may be given to μ; or if a special number of lines be required, a value may be found to suit. Thus let L be the greatest value that is to be given to l, and m the number of lines required on each side of O. We shall have

$$\frac{a + L}{a - L} = \mu^m, \text{ or } \log \mu = \frac{1}{m} \log \frac{a + L}{a - L},$$

which determines the value of μ.

The successive values of B D′ are then

$$\frac{2a}{\mu - 1}, \frac{2a}{\mu^2 - 1}, \frac{2a}{\mu^3 - 1}, \ldots \frac{2a}{\mu^m - 1}.$$

The successive values of B C are simply alternate values of these, namely

$$\frac{2a}{\mu^2 - 1}, \frac{2a}{\mu^4 - 1}, \ldots \frac{2a}{\mu^{2m} - 1}.$$

Plate IV. shows the equipotential lines drawn for a value of $\mu = \frac{5}{3}$.

22. *Resistance.*—The resistance, in the direction of the flow, of the part of the sheet extending between two given equipotential circles follows directly from equation (4) (in § 19). Thus for the potentials of the circles characterized by the ratios $r_1 : r'_1$ and $r_2 : r'_2$ respectively, where r_1 and r_2 are distances from the source, and r'_1 and r'_2 the corresponding distances from the sink, we have

$$U_1 = \frac{Q}{2\pi\kappa\delta} \log \frac{r'_1}{r_1} \text{ and } U_2 = \frac{Q}{2\pi\kappa\delta} \log \frac{r'_2}{r_2},$$

whence

$$U_1 - U_2 = \frac{Q}{2\pi\kappa\delta} \log \frac{r'_1 r_2}{r_1 r'_2}.$$

Consequently the resistance of the part of the sheet between these circles is

$$R=\frac{U_1-U_2}{Q}=\frac{1}{2\pi\kappa\delta}\cdot\log\frac{r'_1r_2}{r_1r'_2}. \quad . \quad . \quad . \quad (5)$$

If ρ_1 is the radius of the circle which has the greater absolute potential, or the one nearest the source, and ρ_2 is the radius of the circle nearest the sink, the similarity of the triangles A P C and B P C in fig. 2, Plate IV., gives

$$\frac{\rho_1}{a+d_1}=\frac{r_1}{r'_1} \text{ and } \frac{\rho_2}{a+d_2}=\frac{r_2}{r'_2}$$

for circles round the source, and

$$\frac{\rho_1}{a+d_1}=\frac{r'_1}{r_1} \text{ and } \frac{\rho_2}{a+d_2}=\frac{r'_2}{r_2}$$

for circles round the sink. Substituting these values, we get

$$R'=\frac{1}{2\pi\kappa\delta}\cdot\log\frac{\rho_2(a+d_1)}{\rho_1(a+d_2)}$$

if both circles surround the source, and

$$R'=\frac{1}{2\pi\kappa\delta}\cdot\log\frac{\rho_1(a+d_2)}{\rho_2(a+d_1)}$$

if they both surround the sink. In the former case we have $\rho_1<\rho_2$, and in the latter case $\rho_1>\rho_2$; but using ρ_1 for the radius of the smaller circle and ρ_2 for that of the larger, we may write

$$R'=\frac{1}{2\pi\kappa\delta}\cdot\log\frac{\rho_2(a+d_1)}{\rho_1(a+d_2)} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

for the resistance between two equipotential circles surrounding the same pole, whether that pole be a source or a sink.

If the circles surround opposite poles, the resistance becomes

$$R''=\frac{1}{2\pi\kappa\delta}\cdot\log\frac{(a+d_1)(a+d_2)}{\rho_1\rho_2}. \quad . \quad . \quad . \quad (7)$$

In the case of circles round the same pole, if the other pole is infinitely distant, the value of R′ becomes

$$\frac{1}{2\pi\kappa\delta}\log\frac{\rho_2}{\rho_1},$$

which is identical with (1), the value found for the resistance of an annular belt of internal radius ρ_1 and external radius ρ_2; and in fact the case supposed (an infinite distance between source and sink) is physically identical with the case of a single pole in an infinite sheet.

When the circles surround opposite poles, if the radii are equal, the resistance becomes

$$\frac{1}{\pi\kappa\delta}\log\frac{a+d}{\rho} \quad \text{or} \quad \frac{1}{\pi\kappa\delta}\log\frac{a+\sqrt{a^2+\rho^2}}{\rho};$$

or when the common radius is small as compared with a,

$$\mathrm{R}=\frac{1}{\pi\kappa\delta}\log\frac{2a}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

23. The following modified forms of (6) and (7) obtained by means of the relations (§ 20)

$$\frac{a+d}{\rho}=\frac{\rho}{h}, \quad a^2=ll', \quad 2\rho=l-l,$$

may also be noted:

$$\left.\begin{aligned}\mathrm{R}'&=\frac{1}{2\pi\kappa\delta}\log\frac{\rho_1 h_2}{\rho_2 h_1},\\ \mathrm{R}''&=\frac{1}{2\pi\kappa\delta}\log\frac{\rho_1\rho_2}{h_1 h_2},\end{aligned}\right\} \quad . \quad (9)$$

$$\left.\begin{aligned}\mathrm{R}'&=\frac{1}{2\pi\kappa\delta}\log\frac{(\sqrt{l'_1}-\sqrt{l_1})(\sqrt{l'_2}+\sqrt{l_2})}{(\sqrt{l'_1}+\sqrt{l_1})(\sqrt{l'_2}-\sqrt{l_2})},\\ \mathrm{R}''&=\frac{1}{2\pi\kappa\delta}\log\frac{(\sqrt{l'_1}+\sqrt{l_1})(\sqrt{l'_2}+\sqrt{l_2})}{(\sqrt{l'_1}-\sqrt{l_1})(\sqrt{l'_2}-\sqrt{l_2})}.\end{aligned}\right\} \quad . \quad (10)$$

24. If two equipotential circles alone are given and it is required to find the resistance between them, the position of the poles A and B being unknown, the expressions given above for R′ and R″ are inapplicable (unless the circles are so small that their centres sensibly coincide with the poles); for in such cases the values of r, r', a, d, h, and l are all unknown. The distance, however, between the centres of the circles is directly measurable; and the resistance can be expressed in terms of this distance and of the known radii as follows.

Calling the distance between the centres D, we have for circles surrounding the same pole,

$$\mathrm{D}=d_2-d_1=\sqrt{a^2+\rho_2^2}-\sqrt{a^2+\rho_1^2},$$

which gives

$$d_1=\sqrt{a^2+\rho_1^2}=\frac{\rho_1^2-\rho_2^2-\mathrm{D}^2}{2\mathrm{D}}, \quad d_2=\sqrt{a^2+\rho_2^2}=\frac{\rho_1^2-\rho_2^2+\mathrm{D}^2}{2\mathrm{D}},$$

and

$$a=\frac{\sqrt{(\rho_1+\rho_2+\mathrm{D})(\rho_1+\rho_2-\mathrm{D})(\rho_1-\rho_2+\mathrm{D})(\rho_1-\rho_2-\mathrm{D})}}{2\mathrm{D}}.$$

Hence (6) may be written *

$$R' = \frac{1}{2\pi\kappa\delta}\log$$

$$\frac{\rho_2\sqrt{(\rho_1+\rho_2+D)(\rho_1+\rho_2-D)(\rho_1-\rho_2+D)(\rho_1-\rho_2-D)}+\rho_1^2-\rho_2^2-D^2}{\rho_1\sqrt{(\rho_1+\rho_2+D)(\rho_1+\rho_2-D)(\rho_1-\rho_2+D)(\rho_1-\rho_2-D)}+\rho_1^2-\rho_2^2+D^2}. \quad (11)$$

When the equipotential circles surround different poles, we have

$$D = d_1+d_2 = \sqrt{a^2+\rho_1^2}+\sqrt{a^2+\rho_2^2},$$

which gives

$$d_1 = \sqrt{a^2+\rho_1^2} = \frac{D^2+(\rho_1^2-\rho_2^2)}{2D}, \quad d_2 = \sqrt{a^2+\rho_2^2} = \frac{D^2-(\rho_1^2-\rho_2^2}{2D},$$

and a the same as before. Hence (7) may be written

$$R'' = \frac{1}{2\pi\kappa\delta}\log$$

$$\frac{\sqrt{(\rho_1+\rho_2+D)(\rho_1+\rho_2-D)(\rho_1-\rho_2+D)(\rho_1-\rho_2-D)}-\rho_1^2-\rho_2^2+D^2}{2\rho_1\rho_2}. \quad (12)$$

25. If instead of considering the whole extent of the sheet we confine our attention to that part of it which is contained between any two lines of flow (see Plate IV.), its resistance will be given by any of the above formulæ if instead of 2π in the denominator we put the angle, say γ, which the lines of flow forming the boundaries make with each other at either pole; for the spaces between every pair of consecutive flow-lines convey equal currents; and since the difference of potential between their ends is the same for all, they all have the same resistance. Consequently the resistance of a part of the sheet made up of such spaces is inversely proportional to the number of them which compose it. Now, using as previously α to stand for the angle which two consecutive lines of flow make with each other at the poles, the total number of spaces into which the whole sheet is divided is $\frac{2\pi}{\alpha}$, and the number of spaces lying between lines of flow whose angle at the poles is γ, is $\frac{\gamma}{\alpha}$; hence, if R_γ is the resist

* This expression was given in a slightly different form by Gaugain on the authority of Blavier (*Ann. de Chim. et de Phys.* Ser. 3. vol. lxvi. p. 203. 1862); and a demonstration has since been published by the latter (*Journal de Physique*, vol. iii. p. 115, April 1874).

ance of the part of the sheet bounded by these lines, and R the resistance of the whole of the sheet, we have

$$\frac{R_\gamma}{R} = \frac{2\pi}{\gamma}.$$

In the case of two flow-lines which form arcs of the same circle drawn through the two poles, we have $\gamma = \pi$, and therefore

$$R_\pi = 2R.$$

Hence, if a small equipotential circle is drawn round each pole, the resistance of the part of the sheet lying between these circles and contained within any circle drawn through the poles is the same, being equal to twice the resistance of the unlimited sheet between the same two equipotential circles, and also equal to the resistance of as much of the unlimited sheet as would be left after cutting out the portion bounded by any circle through the poles. Consequently if ρ is the common radius of the small equipotential circles, we have, for the resistance of a disk on the edges of which the poles are situated, and also for that of the space outside it, according to §§ 22 & 24,

$$R = \frac{2}{\pi\kappa\delta} \cdot \log \frac{2a}{\rho} \quad \text{or} \quad R = \frac{2}{\pi\kappa\delta} \cdot \log \frac{D}{\rho}. \qquad . \quad (13)$$

26. It may be observed that the method by which the resistance was obtained above (§ 22) is essentially the same as that adopted by Kirchhoff. The method by which we first obtained the expression, however, was founded on the consideration of the equality of the resistance of the curvilinear rectangles into which the sheet is divided by the intersecting systems of lines of flow and equipotential lines (*vide* Plate IV.). This process, which may be stated as follows, we afterwards found to be somewhat similar in principle to that employed by Smaasen.

Let two circular electrodes be placed upon the sheet so as to coincide with two of the natural equipotential circles, which for simplicity we will assume to have the same radius ($=\rho$). The spaces between consecutive lines of flow are of equal resistance, and may for the present purpose be regarded as so many independent conductors connecting the two poles and combined in multiple arc. Consequently the resistance of the sheet which is made up of these spaces, is equal to the resistance of one of them divided by their whole number. These spaces in their turn are each composed of $2n$ equi-resisting rectangles arranged in

series; so the resistance of each space is $2n$ times the resistance of one of the rectangles, $2n$ being (as is evident from Plate IV.) the whole number of equipotential lines in the sheet which are not obliterated by the electrodes.

Now consider the resistance of one of the four rectangles at the middle of the figure. Its length O D $=l=a\frac{\mu-1}{\mu+1}$ (§ 20) (for, $n=0$ giving the straight equipotential line through O (§ 19), the next one will be given by $n=1$). Its breadth b, which in fig. 3 is denoted by O Q, will be

$$a\cot\frac{\pi-\alpha}{2}=a\tan\frac{\alpha}{2}.$$

Calling δ the thickness and κ the conductivity of the sheet as before, the resistance of the rectangle will be approximately

$$R_0=\frac{l}{b\kappa\delta}=\frac{1}{\kappa\delta\tan\frac{1}{2}\alpha}\cdot\frac{\mu-1}{\mu+1}.$$

The number of equipotential lines between the electrodes is $2n$ if the electrodes are of equal size and coincide with the nth equipotential circle on each side of the middle line. The radius of this circle is given in § 20 as

$$\rho=\frac{2a\mu^n}{\mu^{2n}-1};$$

solving which we get

$$\mu^n=\frac{a\pm\sqrt{a^2+\rho^2}}{\rho}.$$

The positive sign only is admissible, since μ is essentially positive; whence the number of lines

$$2n=\frac{2}{\log\mu}\log\frac{a+\sqrt{a^2+\rho^2}}{\rho}.$$

So, then, the resistance of a strip between two consecutive flow-lines is

$$R_1=2nR_0=\frac{2}{(\mu+1)\kappa\delta\tan\frac{1}{2}\alpha}\cdot\frac{\mu-1}{\log\mu}\log\frac{a+\sqrt{a^2+\rho^2}}{\rho}.$$

The number of flow-lines in the sheet is (§ 12)

$$\frac{2\pi}{\alpha};$$

therefore the resistance of the whole sheet is approximately

$$R=\frac{\alpha}{2\pi}R_1=\frac{1}{(\mu+1)\pi\kappa\delta}\cdot\frac{\alpha}{\tan\frac{1}{2}\alpha}\cdot\frac{\mu-1}{\log\mu}\log\frac{a+\sqrt{a^2+\rho^2}}{\rho}.$$

Now, make the rectangles infinitely small (and therefore rectilinear), by letting α approach 0 and μ approach 1; then

$$R=\frac{1}{\pi\kappa\delta}\log\frac{a+\sqrt{a^2+\rho^2}}{\rho}.$$

the same expression as that obtained at the end of § 22 for the same case.

The resistance of the segment contained between two flow-lines intersecting each other with the angle γ is, under the same circumstances (§ 25),

$$R_\gamma=\frac{2}{\gamma\kappa\delta}\log\frac{a+\sqrt{a^2+\rho^2}}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Any number of Poles in an Infinite Sheet.

27. *Lines of Flow.*—The equation to the flow-lines for two equal and opposite poles, A and B, in an unlimited sheet, was obtained in § 11 in the following form,

$$\angle APB=n\alpha,$$

where n is a parameter whose value changes by unity on passing from one line to the next, and α is the angle between adjacent rays of the equiangular pencil of flow-lines produced by either pole separately. If the straight line AB be taken as the axis of x, the above equation may be written

$$\angle APB=\angle PBX-\angle PAX=n\alpha.$$

Moreover it was shown, in § 10, that the system of flow-lines resulting from the composition of any two similar systems is obtained by drawing lines through the alternate angles of the quadrilaterals produced by the mutual intersection of the lines of the component systems, in directions concurrent with both the flow-lines which intersect each other at each angle. Hence, if the poles A and B (Pl. IV. fig. 1) were of the same sign (both sources or both sinks), the lines of flow of the resultant system, instead of passing through the points P, Q, R and P_1, Q_1, R_1, &c., as there represented, would pass through the supplementary angles of the quadrilaterals, namely from Q′ to P_1, from R′ to Q_1, &c. It is evident that, as we pass from point to point of a line so

drawn, the angles which the *radii vectores* from the poles make with the axis of x change equally, but in opposite directions; hence this equation may be written

$$\angle \mathrm{PBX} + \angle \mathrm{PAX} = n\alpha.$$

28. If, instead of the line through the poles, any other straight line in the plane of flow be taken as the direction of reference for measuring the angles made by the *radii vectores*, the form in which the equations to the lines of flow have just been given will still be applicable. Thus, if the line X (Pl. V. fig. 1), making any angle δ with A B, be taken as the direction of reference, and if a and b be the points where it is met by P A and P B respectively, it is clear that the angles P a X and P b X are each of them less than the previous angles P A X and P B X by the angle δ, but that for each flow-line they give a constant sum or difference according as the poles are of the same or of opposite kinds. Hence, denoting the angles which the *radii vectores* make with any fixed line by θ_1 and θ_2 respectively, and agreeing to distinguish sources and sinks by a difference of intrinsic sign of the angles corresponding to them, we may write the equation to the flow-lines due to two equal poles, whether similar or opposite, in the general form

$$\theta_1 + \theta_2 = n\alpha.$$

29. In like manner, the lines of flow due to any number of equal poles are represented by the equation $\Sigma\theta = n\alpha$, the angles which the *radii vectores* from the several poles make with any fixed line being respectively denoted by $\theta_1, \theta_2, \ldots$. In order to prove this, it is sufficient to show that, if the expression holds good for any given number of poles, it will still be true for one pole more; for since the formula has been proved for two poles, this would enable us to advance, by successive additions of one pole at a time, to any number.

Let, then, the curves P″ Q′ R, P′ Q R_1, and P Q_1 R_2 (Pl. V. fig. 2) represent parts of three successive lines of flow due to any number k of equal poles, and let them be characterized respectively by the following values of $\Sigma\theta$, namely $(n+1)\alpha$, $n\alpha$, and $(n-1)\alpha$. If there is another pole of the same strength at L, which we will suppose in the first instance to be a *source*, the pencil of flow-lines diverging from it with the common difference of angle α will intersect the flow-lines from the k sources, producing a number of quadrilaterals through whose opposite angles the

lines of flow will pass which result from the action of the pole at L combined with that of all the rest (§ 10). If the direction of the flow along the lines of the component systems be as represented by the arrow-heads* in the figure, one line of the resultant system will pass through the points P′ and Q′, another through P, Q, and R, and a third through Q_1 and R_1. Let the lines joining the source L with the points P, Q, and R cut the fixed line X in p, q, and r respectively; then, for the flow-line through Q due to the k poles, we have by hypothesis

$$\theta_1+\theta_2+\ldots+\theta_k=n\alpha;$$

and for the line through the same point due to the source L we have

$$\angle \mathrm{Q}q\mathrm{X}=m\alpha,$$

and therefore, for both together,

$$\Sigma\theta+\mathrm{Q}q\mathrm{X}=(n+m)\alpha.$$

For the point P, we have similarly

$$\Sigma\theta+\mathrm{P}p\mathrm{X}=(n-1)\alpha+(m+1)\alpha;$$

and for the point R we have

$$\Sigma\theta+\mathrm{R}r\mathrm{X}=(n+1)\alpha+(m-1)\alpha.$$

Consequently for each of these points, which we already know are situated on the same flow-line of the resultant system, we have

$$\theta_1+\theta_2+\ldots+\theta_k+\theta_l=(n+m)\alpha.$$

In the same way, we should get for the points P′ and Q′ situated on the line of flow of the resultant system which lies next to the line through P, Q, and R on the left, the respective values which follow:

$$\Sigma\theta+\mathrm{P}'p\mathrm{X}=n\alpha+(m+1)\alpha,$$
$$\Sigma\theta+\mathrm{Q}'q\mathrm{X}=(n+1)\alpha+m\alpha,$$

* For the purpose of the argument in the text, we may assume arbitrarily either the *direction of flow* along the assumed system of lines of flow, or the *direction of increase* in the value of n (in $n\alpha$), but not both; for a definite assumption regarding one of these conditions determines the other also. If the system of flow-lines for any given set of poles is built up by successive repetitions for one pole at a time of the construction given in § 10, it will be found that the conventional distinction between positive and negative angles leads to the following rule as to the connexion between the two above-named conditions:—Going along a given flow-line *with the flow*, any line on the *left hand* corresponds to a *higher* value of n than the given line, while a line on the *right hand* corresponds to a *lower* value of n.

and therefore for each of them

$$\theta_1+\ldots+\theta_k+\theta_l=(n+m+1)\alpha.$$

The same reasoning gives for the points Q_1 and R_1 on the line of flow next on the right to that through P, Q, and R,

$$\theta_1+\ldots+\theta_k+\theta_l=(n+m-1)\alpha.$$

30. Next suppose the additional pole at L to be a *sink*; then, since the resultant flow must be everywhere concurrent with both the components, the new flow-lines will pass through the other pair of angles of each quadrilateral; for instance, three consecutive lines will pass through the points P′ and Q_1, through P″, Q, and R_2, and through Q′ and R_1; and by applying the same considerations as above, it will be seen that these lines are characterized by the following values of the expression

$$\theta_1+\ldots+\theta_k+\theta_l,$$

namely:—for the line through P′ and Q_1,

$$(n-\overline{m+1})\alpha;$$

for that through P″, Q, and R_2,

$$(n-m)\alpha;$$

and for that through Q′ and R_1,

$$(n-\overline{m-1})\alpha.$$

Hence it follows that sources and sinks affect the value of the sum of angles denoted by $\Sigma\theta$ in opposite ways, and therefore that, as assumed provisionally above (§ 28), sources and sinks must be distinguished by a difference in the intrinsic sign of the angles which the *radii vectores* drawn from them make with the fixed line. When this is done, the equation

$$\Sigma\theta=n\alpha$$

is true for the lines of flow produced by any number of equal poles, whether of the same sign or not.

Further, if we have poles of unequal strengths supplying (or removing) quantities of electricity in the unit of time denoted, say, by $Q_1, Q_2, \ldots$, these may be regarded as each of them produced by the coalescence of a corresponding number of equal poles supplying in unit of time a quantity of electricity q which is taken small enough to be a common measure of all the quantities $Q_1, Q_2, \ldots$. In applying the general formula to such a case, the value of θ corresponding to each pole would have to be taken a number of times equal to the number of constituent

poles of strength q required to make up the actual pole. Hence

$$\Sigma(Q\theta)=Q_1\theta_1+Q_2\theta_2+\ldots=n\alpha \quad . \quad . \quad . \quad (15)$$

is a perfectly general expression for the lines of flow produced by any number of poles of any strength.

31. *Equipotential Lines.*—It was shown in § 17 that the system of equipotential lines resulting from the composition of any two systems, for which the constant difference of potential on passing from one line to another is the same, is given by drawing lines through the alternate angles of the quadrilaterals formed by the mutual intersection of the lines of the component systems, if the angles are taken in such order that on going from any one to the next we pass, in one of the component systems, to a line of higher, and in the other to one of lower potential. We know also (see § 5) that the equipotential lines due to a single pole are concentric circles whose radii vary according to the terms of a geometrical progression, the potential increasing in the case of a source as the radii decrease, and, in the case of a sink, as the radii increase. Hence it follows very simply, by the application of reasoning exactly analogous to that employed in § 29, that if P be a point on an equipotential line due to any number of equal sources at the points A, B, C, ..., and sinks at the points A′, B′, ...,

$$\frac{AP\,.\,BP\,.\,CP\ldots.}{A'P\,.\,B'P\ldots.}=\mu^n=\text{constant},$$

where μ is the constant ratio of the radii of consecutive equipotential circles due to a single pole, while n is a number characteristic of the particular line on which the point P is situated and increasing by unity if this point passes from any given line to the line of next lower potential. For shortness we may put r_1 for AP, r_2 for BP, ..., and write the above equation thus,

$$\frac{r_1\,.\,r_2\,.\,r_3\,.\ldots.}{r'\,.\,r'_2\,.\ldots.}=\mu^n,$$

or

$$\Sigma\log r=n\log\mu.$$

32. No part of the above reasoning will be disturbed if we suppose that each of the equal poles hitherto discussed emits (or absorbs) in unit of time a quantity of electricity equal to unity; and as this supposition will simplify the consideration of poles

of unequal strength, we will adopt it. Then, when the poles are unequal, let them severally emit in unit of time quantities of electricity represented respectively by Q_1, Q_2, Q_3, ... (the absorption of electricity by sinks being reckoned as negative emission), the effect will be the same as if Q_1, Q_2, ... poles of unit strength coincided at the points occupied by the actual poles. Hence, in such a case, the *radii vectores* from the several poles must be taken Q_1, Q_2, ... times in the above formula, and it thus becomes

$$\left.\begin{aligned} r_1^{Q_1}.r_2^{Q_2}.r_3^{Q_3}\ldots. &= \mu^n \\ \Sigma(Q\log r) &= n\log\mu, \end{aligned}\right\} \quad \text{or} \quad \ldots\ldots \quad (16)$$

which is a quite general expression for the equipotential lines due to any distribution of poles of any strength.

33. To find the actual potential at any point due to a given set of poles, we must recur to equation (2), § 7. Let V_1, V_2, ... be the potentials which the various poles would produce at the given point if each acted separately, and let ϕ_1, ϕ_2, ... be the potentials produced by each separately at unit distance from itself; then, taking the other symbols in the senses already defined, we have

$$V_1=\phi_1-\frac{Q_1}{2\pi\kappa\delta}.\log r_1,$$

$$V_2=\phi_2-\frac{Q_2}{2\pi\kappa\delta}.\log r_2;$$

$$\vdots \qquad\qquad \vdots$$

and for the resultant potential at the given point,

$$U=\Sigma V=\Sigma\phi-\frac{1}{2\pi\kappa\delta}.\Sigma(Q\log r). \quad \ldots \quad (17)$$

Since, for a given set of poles, $\Sigma\phi$ is constant, this gives for a locus of constant potential

$$\Sigma(Q\log r)=\text{const.}=2\pi\kappa\delta(\Sigma\phi-U),$$

which is in accordance with (16); for we may write

$$\Sigma\phi-U=n\Delta v,$$

and by § 8, when $Q=1$, we have $\log\mu=2\pi\kappa\delta.\Delta v$.

34. As already observed (§ 1), both the general equations

$$\Sigma(Q\theta)=\text{const.} \quad \text{and} \quad \Sigma(Q\log r)=\text{const.}$$

were given thirty years ago by Kirchhoff, who obtained the

former as a purely mathematical consequence of the latter. In fact, as the systems of lines represented by them are orthogonal to each other, it follows that their first derived functions differ only in one having $\frac{dy}{dx}$ where the other has $-\frac{dx}{dy}$; hence differentiation and subsequent integration, after making this substitution, convert one expression into the other*, This relation between the two equations is of importance, since the form of the equipotential lines can be readily determined experimentally, whereas no practicable method exists for ascertaining experimentally the course of the lines of flow.

35. Before going further we may point out some general properties of lines of flow and equipotential lines. Physically considered, a line drawn from any one pole to any other, so that no electricity crosses it at any point, may be regarded as an independent line of flow, as was done in treating the case of two equal and opposite poles. The lines, however, given by the equation $\Sigma(Q\theta) = \text{const.}$ are continuous curves each of which passes through every pole; and, in a mathematical sense, the whole of the curve given by any one value of the constant must be regarded as a single line of flow. When flow-lines are spoken of in the sequel, it is to be understood that they are *complete* in the above sense. If all the poles are of equal strength, every flow-line passes once through each of them; hence, if they are unequal, every line passes through each of them respectively as

* Let the coordinates of the several poles be (a_1b_1), (a_2b_2), &c. The distances of a point from each will be $r_1=\sqrt{(x-a_1)^2+(y-b_1)^2}$, $r_2=$ &c., and the angles which the joining lines make with the axis of x will be given by $\tan\theta_1=\frac{y-b_1}{x-a_1}$, $\tan\theta_2=$&c. Then

$$\frac{d}{dx}\Sigma(Q\log r)=\Sigma\,.\,Q\frac{d}{dx}\log\sqrt{(x-a)^2+(y-b)^2}$$

$$=\Sigma\,.\,Q\left\{\frac{x-a}{(x-a)^2+(y-b)^2}+\frac{y-b}{(x-a)^2+(y-b)^2}\cdot\frac{dy}{dx}\right\}.$$

Now make the substitution, equivalent to writing $\phi+\frac{\pi}{2}$ for ϕ, and integrate,

$$\int\Sigma\,.\,Q\left\{\frac{y-b}{(x-a)^2+(y-b)^2}-\frac{x-a}{(x-a)^2+(y-b)^2}\cdot\frac{dy}{dx}\right\}dx$$

$$=\Sigma\,.\,Q\tan_{-1}\frac{y-b}{x-a}=\Sigma(Q\theta).$$

often as it contains the greatest common measure of the strengths of the whole set. Moreover, if on any flow-line there is a point of no flow, the line will intersect itself at this point one or more times. Separate flow-lines cannot intersect each other anywhere except at a pole.

Equipotential lines form closed curves which always surround one or more poles. At points where the strength of the current is nothing, the same equipotential line cuts itself once or oftener; in such cases there is at least one pole within each loop formed by the line. Separate equipotential lines never intersect each other.

The method of superposition, by which any system of flow-lines or equipotential lines can be drawn, is easily carried out by drawing, in rather strong lines on white paper, one of any two systems that are to be compounded to a resultant system, and drawing the other component system on tracing-paper. Placing the two drawings one over the other in any required manner, and laying a second piece of tracing-paper on the top, the curves which pass through the intersections of the two component systems can be drawn at once. The resultant system may then, if required, be recompounded with one of the components by the same process.

36. The general course of the lines of flow and equipotential lines for a few of the simpler cases is shown in Plate V.

Two equal similar poles (fig. 3).—The equation to the lines of flow for this case is

$$\theta_1+\theta_2=n\alpha,$$

which is equivalent to

$$x^2-y^2-2xy\cot n\alpha=a^2,$$

if the straight line through the poles is taken as the axis of x, and its middle point as the origin. They are a system of rectangular hyperbolæ cutting each other at the poles with a constant difference of angle ($=\alpha$). The value $n\alpha=\pi$ gives the two axes, the origin where they intersect being a point of no flow. The equipotential lines for the same case are a system of lemniscates, $r_1r_2=\mu^n$, and are identical with the loci of equal flow for two equal opposite poles (§ 15). The strength of the current at any point at a distance r from the origin is $s=\frac{Q}{\pi}\cdot\frac{r}{r_1r_2}$; and consequently the loci of constant flow are the same as the equipotential lines

for three equal poles—two of the same sign, and one of the opposite sign halfway between them, as shown in fig. 4. The strength of the current on the self-cutting curve is $\frac{Q}{2\pi a}$. For all other values of s the curve consists of two separate branches; when s is $> \frac{Q}{2\pi a}$, one branch encloses each pole; and when s is $< \frac{Q}{2\pi a}$, one branch surrounds both poles and the other surrounds neither, being within the central loop of the self-cutting curve. Each of the loci cuts any straight line through the origin harmonically with respect to its intersections with the circle $r=a$.

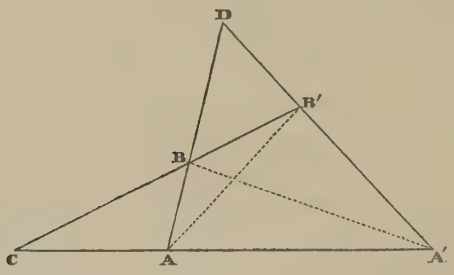
Three equal poles.—A special case coming under this head, which admits of some important applications, is that in which there are three equal poles in a straight line, one of the outside ones being of opposite sign to the other two—for instance, two sources A and A′, and one sink C. In this case the flow-line given by $\theta_1+\theta_2-\theta_3=\pi$ consists of two branches, one of which is the straight line passing through the poles; while the other is a circle whose centre is the sink C (fig. 5), and whose radius C P is a mean proportional between the distances from the sink to the two sources respectively (or $CP=\sqrt{CA \cdot CA'}$). If the two sources coincide, the systems of lines of flow and equipotential lines become what are shown in fig. 6, and the circular branch of the flow-line $n\alpha=\pi$ passes through the pole of double strength. In accordance with what was said above (§ 34), every flow-line of the system passes twice through the double source and once through the sink.

Four equal poles.—Four equal poles of the same sign situated at the corners of a square give the system of flow-lines represented in fig. 7. The two diagonals of the square and the two straight lines through the middle point parallel to the sides of the square are given equally by the values $n\alpha=\pi$ and $n\alpha=0$. For $n\alpha=\frac{1}{2}\pi$ we get a curve of four branches, one of which is situated symmetrically in each quadrant.

A special case of four equal poles, which is important in consequence of its being susceptible of experimental verification, is presented by a combination of two sources and two sinks arranged in a manner that may be regarded as a duplication, with inversion of signs, of the system of three poles shown in fig. 5. Let

the pole at C be a sink, those at A and A′ being sources. Let two additional sinks be placed at B and B′ (fig. 8), points on another straight line through C, and let a source equal to them be put at C; then the three poles of this new system would give by themselves, as one of their flow-lines, the straight line C B and the circle with centre C and radius $= \sqrt{CB \cdot CB'}$. If, however, the three new poles be made equal to the three poles of the first set, the sink originally at C and the equal source now put there will exactly compensate each other, and the combined system will be reduced to two sources at A and A′ and two sinks equal to them at B and B′. If, further, the points B and B be taken so that $CB \cdot CB' = CA \cdot CA' = CP^2$, the circle of radius CP

Fig. 8.



will be a common line of flow in each of the systems due to C, A, and A′ and to C, B, and B′, and will therefore remain a line of flow in the resultant system (see fig. 9) due to A, A′, B, and B′. Moreover, since these four points are placed so that one circle can be drawn through them all, it is easy to see that this circle will be another flow-line of the resultant system; for the four poles may be grouped in two ways into two pairs of equal opposite poles (either A, B and A′, B′, or A, B′ and A′, B), each of which would separately give this circle as a line of flow. If the straight lines AB and A′B′ are drawn and produced to their intersection at D (fig. 8), a circle drawn about this point as centre with radius $= \sqrt{DB \cdot DA} = \sqrt{DB' \cdot DA'}$ will be an equipotential line common to the systems due respectively to the source and sink at A and B, and to those at A′ and B′; consequently this circle is also an equipotential line of the resultant system due to all four poles. Another equipotential line of the resultant system would

be the circle (in this case imaginary) drawn with the point of intersection of AB′ and A′B as centre so as to cut each of these straight lines in points harmonically situated with respect to their extremities.

Again, if four equal poles be taken situated as in the last example, but so that poles of the same sign are diagonally opposite each other—in other words, if the signs of either A and B or of A′ and B′ be interchanged, the circle passing through the four poles will still be a flow-line of the resultant system, but C, as well as D, will now be the centre of an equipotential circle, while the imaginary circle will be a line of flow.

37. *Resistance.*—Equation (17) gives, for the difference of potential of any two points whose respective distances from the several poles are $r_1, r_2, r_3, \ldots$, and $r'_1, r'_2, \ldots$ the value

$$\begin{aligned} U - U' &= \frac{1}{2\pi\kappa\delta}\left[\Sigma(Q \log r') - \Sigma(Q \log r)\right] \\ &= \frac{1}{2\pi\kappa\delta}\Sigma\left(Q \log \frac{r'}{r}\right), \end{aligned}$$

since, for each pole, there are the corresponding terms $Q_n \log r_n$ and $Q_n \log r'_n$. If there are altogether K poles, k being of one sign and k' of the opposite sign, where k is not less than k', and if, further, they are all of the same strength, the quantity of electricity crossing each complete equipotential line in unit of time is kQ; consequently the resistance of the portion of the sheet lying between the equipotential lines which pass through the given points is

$$R = \frac{U - U'}{kQ} = \frac{1}{2\pi\kappa\delta \cdot k} \cdot \Sigma\left(\log \frac{r'}{r}\right). \quad . \quad . \quad (18)$$

This formula is in principle quite general; but the practical application of it in actual cases requires that we should know the position of the poles from which the distances $r_1, r_2, \ldots$ are to be measured; and these cannot be ascertained (or at least not by elementary methods), except for comparatively few and simple cases; for although it is comparatively easy to determine the equipotential lines for a given set of poles, the inverse problem, of finding the distribution of poles required to produce equipotential lines coinciding with two given curves on the conducting sheet, presents in general very great mathematical difficulties, and has hitherto received only partial solutions.

38. We will give here, in the first place, the approximate application of the general formula to the case of two sources and two equal sinks at the angles of a quadrilateral inscribed in a circle and so placed that unlike poles are diagonally opposite each other. This is the arrangement shown in figs. 8 and 9, where A and A′ may be taken as sources and B and B′ as sinks. It is evident that the equipotential lines very near the poles will each consist of two branches, one of them surrounding one source (or sink), and the other surrounding the other source (or sink), and also that they will be very approximately circles having the poles at their centres. Hence, if the sources and sinks are formed by four circular electrodes, whose common radius ρ is a small fraction of the distance between any two of them, and if they are so placed that the distances of their centres A, A′, B, and B′ from a fixed point C fulfil the condition $CA \cdot CA' = CB \cdot CB'$ (which is equivalent to saying that their centres are on the circumference of one and the same circle), we may without serious error regard the circles of contact between the electrodes and the conducting sheet as forming together a pair of equipotential lines due to such a distribution of poles as that referred to at the beginning of this paragraph. Then, taking points on the circumference of the circles round A and B as the points to which the values of r and r' in equation (18) respectively refer, we have approximately

$$\Sigma\left(\log\frac{r'}{r}\right)=\log\frac{(AB)^2 \cdot AB' \cdot A'B}{\rho^2 \cdot AA' \cdot BB'};$$

and since for the case supposed $k=k'=2$, the resistance of the part of the sheet extending between the pair of circles round A and A′ and the pair round B and B′ is represented with similar approximation by

$$R=\frac{1}{4\pi\kappa\delta}\cdot\log\frac{(AB)^2 \cdot AB' \cdot A'B}{\rho^2 \cdot AA' \cdot BB'} \quad . \quad . \quad . \quad . \quad (19)$$

If a circle be drawn with the centre C and radius

$$CP=\sqrt{CA \cdot CA'}=\sqrt{CB \cdot CB'},$$

this circle will coincide, as already pointed out (§ 35), with a flow-line of the system due to the combined action of the four poles. Consequently no electricity passes into or out of this circle; and therefore the whole of the electricity supplied by the source A flows to the sink B *inside* the circle (fig. 9), while the whole of what is supplied by the source A′ flows to B′ *outside*

the circle. It follows, since the sources A and A were assumed of the same strength, that the resistance offered by the part of the sheet lying within the circle of radius CP to the flow of electricity between the electrodes A and B is the same as the resistance of the part of the sheet lying outside this circle to the flow of electricity between the electrodes A′ and B′. Hence also the resistance of a disk bounded by the circle in question and containing the two electrodes A and B is equal to *twice* the resistance of the entire sheet to all four poles, and is therefore represented by

$$R' = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{(AB)^2 \cdot AB' \cdot A'B}{\rho^2 \cdot AA' \cdot BB'}, \quad . \quad . \quad . \quad (20)$$

which is the formula referred to in § 1 (page 115) as having been given by Kirchhoff for the resistance of a circular disk with two small circular electrodes anywhere upon it.

39. As already stated, this formula is only approximate, and in certain special cases it entirely fails. For instance, if one (or both) of the poles passes to the edge of the disk, then, in order that the circumference may still continue to be a line of flow, the second pole of the same sign must coincide with it; consequently in such a case AA′ or BB′, or both, will vanish, and the expression for the resistance fails by becoming infinite. The reason evidently is that equation (20) was got by assuming ρ to be very small in comparison with any of the distances between the poles; and this can no longer be true when any two of them coincide. By slightly modifying the notation, we can obtain an expression which does not fail in the same way. Thus, let the points to which the values of r_1, r_2, ... refer be respectively a point F where the straight line A B cuts the equipotential circle round A, and a point G where the same line cuts the equipotential circle round B. Then we have AF = BG = ρ, and the expression for the resistance of the sheet to the current from all four poles becomes

$$R = \frac{1}{4\pi\kappa\delta} \cdot \log \left(\frac{BF \cdot B'F}{AF \cdot A'F} \cdot \frac{AG \cdot A'G}{BG \cdot B'G} \right), \quad . \quad (19a)$$

while the resistance of the disk bounded by the circular flow-line may be written

$$R' = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{AG \cdot BF \cdot A'G \cdot B'F}{\rho^2 \cdot A'F \cdot B'G}. \quad . \quad . \quad (20a)$$

In this form the expression for R′ admits of extension to some

special cases. For example, let the pole A pass to the edge of the disk, while B goes to the centre; then A′ will coincide with A, and B′ will go to an infinite distance. We have then sensibly AG=BF=A′G= the radius of the disk (=P), also A′F=ρ, and the resistance to the flow between the pole at the edge and the pole at the centre of the disk becomes

$$\frac{3}{2\pi\kappa\delta} \cdot \log \frac{P}{\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If both poles go to the edge, A′ coincides with A, and B′ with B, the system being reduced to two equal opposite poles at a distance AG=BF=2a from each other. In accordance with this, the resistance of the whole sheet becomes

$$\frac{1}{\pi\kappa\delta} \cdot \log \frac{2a}{\rho},$$

which is the value already given for the same case in equation (8).

40. If, all else remaining as before, the signs of the poles A′ and B′ are interchanged (that is, if A′ becomes a sink and B′ a source), the circle drawn with the centre C and radius $\sqrt{CA \cdot CA'}$ becomes an equipotential line instead of a line of flow, the circle with centre D and radius $\sqrt{DB \cdot DA}$ remains an equipotential line, and the only circular lines of flow are the circle through the four poles and the imaginary circle (§ 36) whose centre is the intersection of AB′ and A′B. The resistance of the sheet to the flow of electricity between this new arrangement of poles is given by interchanging in (19a) the signs of the *radii vectores* drawn from A′ and B′, whereby we get

$$R = \frac{1}{4\pi\kappa\delta} \cdot \log \frac{AG \cdot BF \cdot A'F \cdot B'G}{\rho^2 \cdot A'G \cdot B'F}. \qquad (19b)$$

The equipotential circles with the centres C and D being common to the system due to the source at A and sink at A′, and also to that due to the sink at B and source at B′, and having equal but opposite potentials when taken as belonging to either of these systems separately, will, in the system due to the four poles, form the two branches of the line of zero potential. The other equipotential lines of the system due to the four poles consist also each of them of two branches, both of which never lie within the same one of the two circles in question. From this it follows that each circle divides the sheet into two equi-

resisting portions; and consequently the resistance of each of them is

$$R'' = \frac{1}{2\pi\kappa\delta} \cdot \log \frac{AG \cdot BF \cdot A'F \cdot B'G}{\rho^2 \cdot A'G \cdot B'F} *. \quad . \quad . \quad (22$$

41. Since half the lines of flow due to each pole lie within the circle drawn through them all, the resistance of the portion of the sheet bounded by this circle is (§ 25) twice the resistance of the unlimited sheet; consequently it is equal to R′ or to R″ according to the arrangement of the poles.

42. It was pointed out in § 38 that the resistance of the part of the sheet outside the circular flow-line with centre C, to the flow from the source A to the sink B, is the same as the resistance of the part inside this circle to the flow from A to B. Accordingly the value of equation (20) remains unaltered when A′ and B′ are put for A and B, and *vice versâ*, and at the same time ρ^2 (or $\rho_A \rho_B$) is replaced by $\rho_{A'} \rho_{B'} = \rho_A \rho_B \left(\frac{A'B'}{AB}\right)^2$. The circles round the two sources then coincide approximately with the two branches of a single equipotential line; and the same is true for those surrounding the sinks.

Similar remarks are applicable to equation (22).

Experimental verifications of some of the conclusions here arrived at will be given in Part II. of this communication.

XVI. *The Electrolysis of certain Metallic Chlorides.* By J. H. Gladstone, *Ph.D., F.R.S., Fullerian Professor of Chemistry in the Royal Institution, and* Alfred Tribe, *F.C.S., Lecturer on Chemistry in Dulwich College.*

We have previously shown that nitrate of copper brought into tension by silver and copper in conjunction is decomposed by free oxygen in solution†. Thinking that chlorine

* It may be noted that by adding together the values of R′ and R″ we get $\frac{1}{2\pi\kappa\delta} \cdot \log \left(\frac{AG \cdot BF}{\rho^2}\right)^2$, which, written in the simpler form

$$\frac{2}{\pi\kappa\delta} \cdot \log \frac{AB}{\rho},$$

is the resistance of a circular disk on whose edge the poles A and B are placed (see equation 13).

† Proc. Roy. Soc. vol. xx. p. 290.

might be substituted for oxygen, we commenced some experiments, employing chloride of copper, and observed some facts which seemed to have an interest from their bearing on the causation of galvanic action.

It is known that if metallic copper be placed in a solution of cupric chloride, it will slowly become covered with a crystalline deposit of the insoluble cuprous chloride:—

$$Cu + CuCl_2 = 2\,CuCl.$$

We found that when metallic copper and platinum are connected by a wire and immersed in cupric chloride, the insoluble salt forms not only upon the copper, but also on the platinum plate, as a white crystalline body. This deposit may generally be observed in about two minutes when the plates are three quarters of an inch apart. The formation of cuprous chloride upon the platinum plate takes place about equally rapidly in solutions containing 2·5 or 10 per cent. of salt. With a 20 per cent. solution the deposit was smaller, and with 40 per cent. practically *nil*, although there was abundant formation of cuprous chloride upon the *copper* plate.

We satisfied ourselves that the action took place equally well in solutions from which oxygen had been rigidly excluded, and also that a current passed from the copper to the platinum through the liquid—that is, from the metal of higher to that of lower potential.

In order to test whether this electrolysis of cupric chloride into CuCl and Cl could be effected by weak currents *ab extra*, we tried the effect of a zinc-platinum cell excited by common water and with platinum electrodes, and found that cuprous chloride deposited upon the negative electrode and chlorine at the positive, a little of which entered into combination with the platinum, but the greater part passed into the liquid. A cell excited with dilute sulphuric acid acted in a similar manner. A single Grove's cell gave for the first two or three minutes cuprous chloride on the negative platinum electrode, but afterwards metallic copper, while chlorine always formed at the positive plate.

As zinc immersed in a salt of copper is capable of throwing down that metal, an experiment was tried with plates of zinc and platinum in connexion immersed in the chloride; the result was a more energetic action than with a copper-platinum couple similarly arranged, and besides a thick coat of cuprous chloride the

edges of the platinum were incrusted with metallic copper. A similar magnesium-platinum couple gave a similar result, but with a decidedly greater proportion of metallic copper.

As there are two chlorides of mercury, similar to the two chlorides of copper, analogous experiments were tried with solution of corrosive sublimate.

A small bell-jar which terminated in a corked orifice was inverted and partly filled with mercury, over which was poured a solution of mercuric chloride; a wire passed from the mercury through the cork to a plate of platinum which hung in the solution, without, however, touching the mercury. On pouring the mercuric chloride upon the mercury, the metallic surface was at once dimmed by a film of mercurous chloride, which increased in quantity; and in the course of an hour or two the insoluble chloride appeared also on the platinum plate, and in twenty-four hours it was sufficiently thick to permit of its removal. On repeating this experiment with gold instead of platinum, the same mercurous chloride was deposited; but at the same time the gold plate was amalgamated, showing that the reduction of the mercuric chloride had not stopped at the first stage, but had actually proceeded to the separation of the metal itself. A similar gold plate immersed in the same solution of corrosive sublimate, but not in connexion with the mercury, showed no trace of deposit or amalgamation; and it was ascertained that gold alone has no power of decomposing moistened mercurous chloride.

An experiment was made with a current *ab extra*. A cell of Grove's was found to decompose mercuric chloride with the formation of the mercurous compound at the negative platinum electrode, while chlorine was given off at the positive one. This is in unison with what was found in the case of the copper salts.

That this action does not depend on the insolubility of the *-ous* chlorides was proved by the behaviour of the iron salts. Ordinary metallic iron is capable of reducing the ferric to the ferrous salt at the common temperature; thus:—

$$2\,FeCl_3 + Fe = 3\,FeCl_2.$$

Platinum does not effect such a reduction; but when the iron is connected with platinum the change takes place more rapidly, and the reduced salt forms also on the negative metal. This, of course, does not render itself evident by any deposit, nor by any appreciable change of colour at first; but if the platinum plate

be lifted out of the solution, and the liquid clinging to it be allowed to drain on to a paper moistened with some ferridcyanide of potassium, Turnbull's blue is the result. Or if a few drops of the ferridcyanide be mixed with the solution of the ferric chloride, on the junction of the iron and platinum the blue colour makes its appearance against each metal. Of course care was taken that the original salt contained no acid. The solution employed was one of 3·5 per cent. in strength.

If plates of magnesium and platinum be immersed in ferric chloride, metallic iron quickly makes its appearance on the platinum plate.

With a weak external battery and platinum electrodes the salt was resolved into chlorine and ferrous chloride, but with a strong battery into chlorine and iron—thus affording another instance of the close analogy between the primary electrolysis produced in the cell itself, and the secondary electrolysis that may be produced by an external battery.

May 22, 1875.

XVII. *A new Polariscope.* *By* Professor W. G. ADAMS.

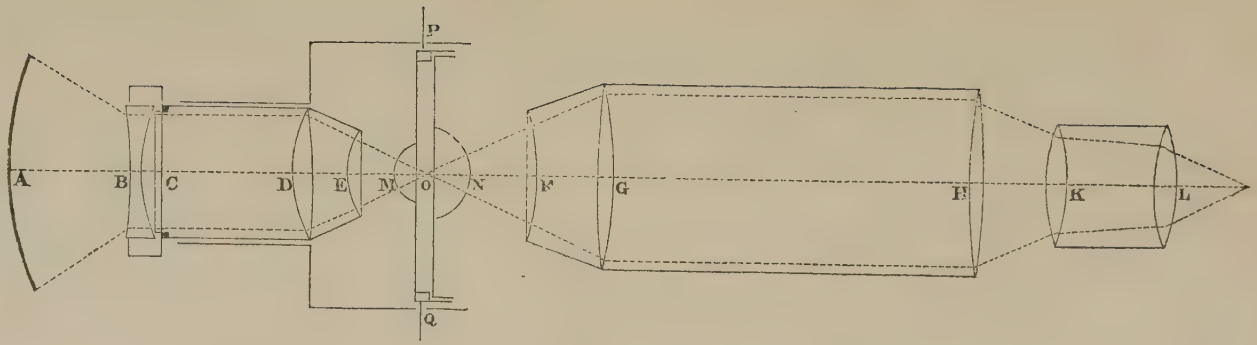
IN devising this instrument, the principal objects in view have been:—

(1) To obtain an extensive field of view.

(2) To afford a means of measuring the rings and the angles between the optic axes of biaxal crystals.

(3) To have a means of immersing the crystal in a liquid in those cases in which the optic axes are too far apart to be seen in air.

These advantages have been obtained by modifying the positions and focal lengths of the lenses usually employed in table polariscopes, so that the rings of a crystal are best seen when there is a space of $1\frac{1}{4}$ inch between the two lenses, one on either side of the crystal. Into this space is introduced a central piece, consisting of a circular box with deep plano-convex lenses fixed, one in the bottom and the other in the top of the box, in such a position that their curved surfaces have a common centre of cur-



A. Concave mirror.
B. Double-concave lens 1 inch in diameter.
C. Tourmaline or other polarizer.
D. Double-convex crossed lens 1 inch in diameter, $1\frac{1}{4}$ inch focal length.
E. Convexo-plane lens $\frac{4}{5}$ inch in diameter, 1 inch focal length.
F. Plano-convex lens 1 inch in diameter, $1\frac{1}{4}$ inch focal length.
G. Double-convex crossed lens $1\frac{1}{2}$ inch in diameter, $1\frac{3}{4}$ inch focal length.
H. Double-convex lens $1\frac{1}{2}$ inch in diameter, 3 inches focal length.
K L. Nicol's prism and eyepiece.
M and N. Portions of hemispherical lenses with O as the common centre of curvature of their spherical surfaces. Radius of curvature of M is $\frac{1}{4}$ inch, and of N is $\frac{3}{8}$ of an inch, and the distance between the lenses M and N about $\frac{1}{8}$ of an inch
P and Q. The ends of the axis supporting the box fixed so that the axis passes through the point O.
The dotted lines show the path of the light through the instrument.

vature, with their flat faces turned towards one another and enclosing the crystal between them. The box can turn about an axis passing through the common centre of curvature.

The form and position of Mirror and Lenses for the Polariscope.

A is a concave mirror about $1\frac{1}{2}$ inch in diameter, such as is ordinarily employed for illuminating microscopes, and mounted in the same manner. A double-concave lens, B, 1 inch in diameter, is placed so as to have its focus at the same point as the principal focus of the mirror. The rays coming from the mirror will then be parallel after passing through the lens. To diminish aberration as much as possible, the radius of curvature of the first face of the lens should be about six times the radius of its second face. The parallel rays then fall on a tourmaline or other polarizer, C, the diameter of which should be nearly equal to that of the lens. They then fall on a double-convex crossed lens D, the first face of the lens having the greatest curvature, so as to diminish the aberration as much as possible. This lens is 1 inch in diameter and $1\frac{1}{4}$ inch focal length.

At a distance of $\frac{1}{4}$ of an inch from this lens is placed a convexo-plane lens E, $\frac{4}{5}$ of an inch in diameter and 1 inch focal length.

These three lenses, B, D, and E, may be fixed in the same piece; and with the above arrangement, rays which are parallel before falling on the mirror will be brought to a focus at a distance of half an inch on the outside of the last lens E. The rays then fall on the central piece M N, consisting of two plano-convex lenses, which are nearly hemispheres, enclosing the crystal between them. The first of these lenses, M, is 5 millims. in thickness, and the radius of its spherical surface is 6·4 millims., or $\frac{1}{4}$ of an inch; the other lens, N, is 7 millims. in thickness, and the radius of its spherical surface is 9·6 millims., or $\frac{3}{8}$ of an inch; and they should be so placed that the two centres of curvature are accurately at the same point O. The distance between them will then be 4 millims., which is very nearly the thickness of most specimens of crystals. If lenses with the same curvatures but of thicknesses $4\frac{1}{2}$ and $6\frac{1}{2}$ millims. respectively be used in place of M and N, there will be a breadth of 5 millims. between them when they are in the best position. This central piece should be so placed that O, the centre of curvature of the two curved surfaces of the lenses, coincides with the focus to which

the rays converge after they come out of the first system of lenses. The rays will then pass through this central piece without having their direction altered, since the crystal will refract the light nearly to the same extent as glass, and the light will pass nearly as if it passed through the centre of a sphere of glass.

After passing through this central piece, the rays diverge from the common centre of curvature and fall on a plano-convex lens, F, 1 inch in diameter, and placed about $\frac{5}{8}$ of an inch from the common centre of curvature. Its focal length should be $1\frac{1}{4}$ inch. At a distance of half an inch from this lens is another lens, G, $1\frac{1}{2}$ inch in diameter and $1\frac{3}{4}$ inch focus, so as to make the rays again parallel. Then at a distance of 3 inches is placed a lens, H, whose focal length is rather less than 3 inches and whose diameter is $1\frac{1}{2}$ inch; and above this the Nicol's prism, K L, with any other apparatus, such as the beautiful arrangements of Mr. Spottiswoode for showing the effects of quartz and other crystals on polarized light. At the principal focus of the last lens, H, should be placed crossed spider-lines; and for accurate measurement a simple lens or eyepiece should be added above the Nicol's prism.

The object of receiving the light from the mirror on a double-concave lens is to avoid aberration and the consequent loss of light; but this involves the use of a large piece of tourmaline. The difficulty is got over in existing polariscopes by employing two convex lenses, causing the rays to cross the axis very near the polarizing tourmaline, and then making them parallel by means of a second convex lens. The central piece may be added to any polariscope provided there is sufficient room between the two systems of lenses to admit the crystal and two lenses, M, N, and to allow of motion about the axis through the common centre of curvature of the surfaces of these two lenses. The two parts, B C D E and F G H, should be so arranged that when the crystal alone is placed at O the rings are best seen; then the crystal should be placed in the box between the two lenses M and N in the bottom and in the cover of the box. This box should be supported on two wires, P and Q, forming an axis passing through O, and should be large enough to contain the cork in which the crystal is placed. The axis is supported in a brass tube, which forms one piece with another brass tube which fits on the piece B C D E. A semicircular disk of brass

with its arc graduated is fixed on the large brass tube about the axis P as a centre; and on this axis is an index for reading-off the angle through which the axis is turned. On turning the box about the axis, no change is produced in the rays which pass through the centre O of the curved surfaces of the lenses; but when a crystal is introduced, the rays may be sent through it in any desired direction on turning the axis, so that either of the optic axes of a biaxal crystal may be brought into the centre of the field. Thus the angle between the optic axes may be measured. The central piece M N is made in the form of a box, so that a liquid may be introduced into it for measurements in those cases in which the optic axes are too far apart to be seen in air.

The advantages to be obtained by the use of this central piece are:—

(1) The extension of the field of view. If the angle in air corresponding to the field of view is 74° without the central piece, then the angle will be increased to about 128° when the central piece is introduced, the central piece giving the same angle in glass that is given without it in air. The field of view may be made to include both optic axes of topaz of Brazil.

(2) When the plane containing the optic axes is at right angles to the axis P Q, either of the optic axes of a biaxal crystal or any ring may be brought into the centre of the field of view where the spider-lines cross one another, and the angles between them accurately measured. Instead of employing a lens for an eyepiece, the centre of the field may be determined by fixing crossed spider-lines at a point halfway between the lenses D and E. These lines will be brought to a focus by the system of lenses E, M, N, F, and G at a distance to the right of G equal to about one tenth of an inch; and at this focus another set of crossed spider-lines should be placed, so that the two sides may by their coincidence determine the centre of the field of view. The spider-lines should be in the focus of the lens H, so that they may be seen on looking through the Nicol's prism.

XVIII. *An Experiment for showing the Electric Conductivity of various forms of Carbon.* *By* H. Bauerman, *F.G.S.*

The following simple method of exhibiting the conducting-power of carbon was brought to my notice by my friend Mr. W. J. Ward, of the Metallurgical Laboratory of the Royal School of Mines, as having been shown to him several years since by Dr. von Kobell, of Munich. As I have not found any account of it published, I have ventured to bring it before this Society.

A fragment of the substance to be tested, whether charcoal, coke, anthracite, or other form of carbon, is held between the jaws of a pair of tongs formed by bending a strip of sheet zinc into a horseshoe form, and immersed in a solution of cupric sulphate. If the carbon is a non-conductor, the copper salt is decomposed, and deposit of copper only takes place on the immersed surface of the zinc; but when it possesses a high degree of conductivity a zinc-carbon couple is formed, and deposit of copper takes place on the surface of the carbon as in ordinary electrotyping.

Of the different forms of carbon experimented upon, the most rapid results have been obtained with some American anthracites, and coals that have been subjected to the action of intruded igneous rocks. The most remarkable of these is an anthracite from Peru, which contains a large amount of sulphur in organic combination, and is found in a nearly vertical position, interstratified in quartzite, in the high plateau of the Andes, about 13,000 feet above the sea-level, near Truxillo. It is probably of secondary age, the metamorphism having taken place at the time of the great trachytic outbursts which form the gold- and silver-bearing rocks of the adjacent mining-district. This is coppered by immersion almost as readily as graphite. The anthracite of Pennsylvania possesses the same property, but not in quite such a high degree. The Heathen coal of South Staffordshire, when altered by the intrusion of the "white-rock" trap, is more slowly coppered; but this is probably due to the resistance interposed by the numerous laminæ of calcite filling the fractures in the mass of the coal, which renders the conductivity less perfect. A specimen of coal from Bengal, altered in the same manner by in-

trusion of igneous rock, behaves much in the same way as coke, being coppered directly. This is rather remarkable, as this coal is a very impure one, and contains such a large quantity of water very intimately combined, probably as a hydrated silicate interspersed through the mass, as to decrepitate explosively when suddenly heated.

The ordinary Welsh anthracite does not appear to be a conductor by this method; but after having been heated to a full red heat it conducts electricity freely. The lowest temperature at which this change takes place appears to be somewhere between the melting-points of zinc (430° C.) and silver (1000° C.), as fragments of anthracite packed in a thin clay crucible and plunged into molten zinc were not found to be altered, but were changed when heated in a bath of melted silver. These limits, although considerably wide apart, are interesting as giving a possible clue to the temperature at which anthracitic metamorphism of coals has been effected in different districts. Mr. W. C. Roberts has recently shown that the alloys of silver and copper have very definite melting-points; it will be possible therefore to determine more nearly the lowest temperature necessary to produce the change.

In the South Wales anthracite district it is well known that no great amount of disturbance has taken place in the position of the coal-seams, while in North America and Peru the change has been accompanied with much more violent action, as evidenced by the greater disturbance of the rocks; and probably a correspondingly higher degree of heat was developed in the mass. The evidence afforded by the coals that have been actually altered by intruded rocks, and must have been highly heated, appears to bear out this view. On the other hand, long-continued exposure to a lower temperature might possibly produce the same effect, and further experiments upon this point would be desirable.

XIX. *On an Apparatus to illustrate the Formation of Volcanic Cones.* *By* C. J. Woodward, *B.Sc.*

A short time ago I spent some weeks in Central France, and had many opportunities of studying the character of the volcanic cones which are so abundant in that district. The many sections of these cones that may be seen at Graveneire and other places, where the scoriæ is obtained for industrial purposes, show their structure clearly, and one can realize the truth and simplicity of the theory accounting for their formation. The theory is so obviously correct that it has not, so far as I know, been subjected to experiment on a small scale. The object of this paper is to describe an apparatus that may be used in the lecture-room to build up model cones and craters.

Soon after my return I tried an experiment at some Iron-works to make such a cone. A large sheet of iron with a hole in the centre was placed over a pipe proceeding from the blast. A quantity of cinders and ashes were supplied to the pipe, and in a short time a cone was built up. For the purpose of lecture-illustration the apparatus about to be described is convenient, and shows very clearly the formation of volcanic cones. It consists of a wooden trough about 18 inches long, with sloping sides; at the bottom of the trough there is a bladed screw to carry forward the ashes, sawdust, or other material used, to an opening through which air from a powerful bellows is forced upward in a vertical stream. A board 3 or 4 feet square with a hole in the centre is placed over the air-jet, so that on turning the screw the sawdust is carried into the stream of air; it is then thrown vertically upward through the hole in the upper board, and on falling down builds up a cone having many of the peculiarities of a natural one. Thus, the angle of slope is almost identical; the sorting of materials is similar, the heaviest portions being near the crater, while the finer particles and dust are carried to a great distance. If, while the miniature eruption is going on, the oxy-hydrogen light be directed horizontally on to the cone, it will be seen that the dust is carried to a distance of many feet. Using alternately sawdust of different colours, the structure of the cone is well seen on making a section of it. This is easily done by cutting vertically through the cone with a sheet of cardboard and sweeping off one half of the mound into a tray. If the jet

of air be vertical and the air of the room is still, the lip of the crater is horizontal; but if the apparatus be near the door or where there is a draught, the materials are carried to one side, and the cone is built up much higher on one side than the other. The same thing, of course, will occur if the jet is not vertical; and as from accidental circumstances the jet frequently changes its direction, a considerable variety of cones is sure to be obtained in a series of experiments.

Birmingham, May 15, 1875.

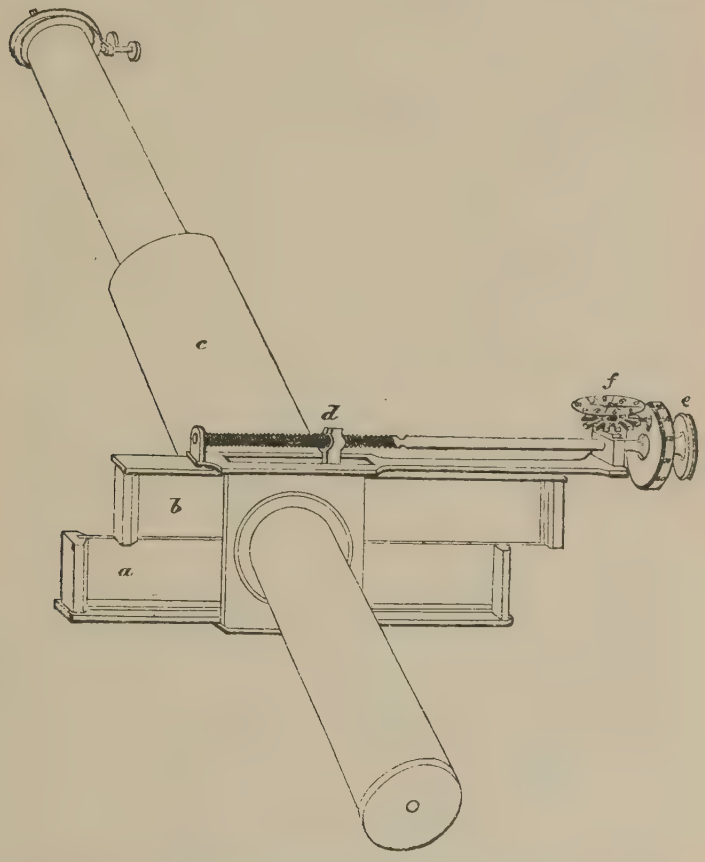
June 26, 1875.

XX. *On a New Form of Micrometer for use in Spectroscopic Analysis.* *By* W. M. WATTS, *D.Sc., Physical-Science Master in the Giggleswick Grammar School.*

IN determining the position of lines in a spectrum by the use of a micrometer-eyepiece or divided arc, it is often difficult to see the cross-wires distinctly without admitting extraneous light, which with faint spectra frequently cannot be done. I have sought to overcome this difficulty by substituting some one known line of the spectrum itself for the cross-wires, and thus to measure the position of unknown lines, by bringing this index-line successively into coincidence with them. Thus, for example, the sodium-line (which is present in nearly every spectrum whether it is wanted or not) may be made to move along under the spectrum, and the displacement necessary to make it coincide with the lines to be measured may be determined by the readings of a micrometer-screw. To accomplish this a convex lens of about 2 feet focus is placed in front of the prism of the spectroscope, between the prism and the observing-telescope, and is divided along a line at right angles to the refracting edge of the prism. One half of the lens is fixed; the other half is made to slide over it by means of a micrometer-screw. When the movable half of the lens is in its normal position, the only effect is to alter the focus of the telescope slightly; but when it is made to slide over the fixed half, the refraction of the prism is

increased or diminished, half of the spectrum appears to move over the other half, and the sodium-line, or any other convenient line of reference, can be brought into coincidence with the lines to be measured.

The figure represents the micrometer as applied to a direct-vision spectroscope of Browning. *a* and *b* are the two strips



cut from a lens of about 2 feet focus, placed in front of the compound prism contained in the body *c*. Of these *a* is fixed; but *b* can be made to slide over *a* by means of a screw *d*; and the displacement is measured by means of the graduated head *e* and the counter *f*.

In order to reduce the indications of this instrument to wave-lengths, I have constructed a series of interpolation-curves from

the data obtained by careful observation of the solar spectrum. One of these curves is for use when the sodium-line ($D_1 = 5889$) is employed as the reference-line, another when the Fraunhofer line b ($b_1 = 5183$) is used, and so on. For use with terrestrial spectra of bright lines, the lithium-, sodium-, and thallium-lines, the hydrogen-lines obtained from a vacuum-tube, or for faint spectra the carbon-lines obtained from a Bunsen burner, furnish convenient reference-lines in sufficient number. The curves are drawn on such a scale that a difference in wave-length of 100 tenth-metres* is represented by four inches, and one turn of the micrometer-screw is represented by one inch.

It requires 21·91 turns of the screw to bring the sodium-line (5892) into coincidence with the thallium-line (5349). The interpolation-curves are very regular and of slight curvature. In fact the readings of this micrometer may be treated in exactly the same way as the readings obtained by means of a graduated arc or bifilar eyepiece, or any of the other forms of measuring-apparatus.

Within the limits allowed by the construction of the instrument, the displacements measured by the micrometer-screw necessary to bring a given reference-line into coincidence with other lines to be measured are proportional to the angles between these lines and the reference-line as they would be obtained by a telescope moving over a graduated arc. The extreme displacement of which the upper half of the lens is capable is 0·69 of an inch; and assuming a refractive index of 1·5 for the glass of which the lens is made, the radius of its surface is 23 inches. The deviation of a ray of light passing through the lens in its extreme position is thus that due to a prism of refracting angle $2 \sin^{-1} \frac{0{\cdot}69}{23{\cdot}00}$; that is, about twice the angle whose sine is 0·03. If, now, we calculate the minimum deviation which a ray of light suffers in passing through the lens at points whose distances from the axis of the lens are represented by 0·01, 0·02, and 0·03 respectively, we obtain angles very nearly in the ratio of these numbers.

The following numbers show the deviation suffered by a ray

* A tenth-metre is $\left(\frac{1}{10}\right)^{10}$ of a metre, or the ten-millionth of a millimetre.

of light in passing through a glass prism ($\mu = 1{\cdot}5$) of varying angle:—

Angle of prism.	Deviation.
$2 \sin^{-1} 0{\cdot}01$	$0^\circ\ 34'\ 22''$
$2 \sin^{-1} 0{\cdot}02$	$1^\circ\ 8'\ 46''$
$2 \sin^{-1} 0{\cdot}03$	$1\ 43\ 12$

The angles $0^\circ\ 34'\ 23''$, $1^\circ\ 8'\ 46''$, $1^\circ\ 43'\ 9''$ would be as the numbers 1, 2, 3.

I add the results of a few measurements made with the new micrometer, designed to test the accuracy of which it is capable. Twenty different readings of the point at which there is coincidence of the lenses gave the following results:—

8·36 8·29 8·35 8·32 8·36 8·34 8·38 8·37 8·34 8·37
Mean 8·35

8·37 8·41 8·26 8·28 8·31 8·35 8·21 8·35 8·38 8·35
Mean 8·33

To bring the sodium-line into coincidence with the thallium-line the following displacements were necessary:—

21·90 21·90 21·89 21·90 Mean 21·90
21·92 21·93 21·94 21·88 Mean 21·92

I have generally taken the mean of four readings as giving the true position of a line.

The light emitted by burning magnesium examined with the spectroscope shows, besides the bright lines corresponding to *b*, a series of bands nearer to F. I had measured the wave-lengths of these bands with the greatest accuracy I could attain by the use of Browning's 6-prism automatic spectroscope in conjunction with a bifilar micrometer-eyepiece. The reference-lines employed were the bright lines of the spark due to zinc, cadmium, iron, and air, and certain Fraunhofer lines. The following results were obtained for the *first* and *second* bands:—

First band. 5006·5 from Fraunhofer lines.
5009·0 from zinc-line 4924
5008·0 from air-lines { 5002 / 5005
5006·5 from Fraunhofer lines.
5007·7 from air-lines { 5002 / 5005

Second band. 4995·0 from Fraunhofer lines.

4997·0 from air-lines $\begin{cases} 5002 \\ 5005 \end{cases}$

4997·4 from air-lines $\begin{cases} 5002 \\ 5005 \end{cases}$

The numbers finally adopted are given below for the whole nine bands; and with them are compared the results recently obtained with the new micrometer. The reference-line employed for these last measurements was the bright line corresponding to *b* present in the same spectrum.

Wave-lengths of Lines of Magnesium Oxide.

Automatic spectroscope with micrometer-eyepiece.	Direct-vision spectroscope with new micrometer.
5007	5006·0
4997	4996·0
4986	4985·5
4975	4974·5
4963	4963·5
4948	4948·5
4934	4934
*4918?	4924
	4914

Each of these results is the mean of five readings. A re-measurement of the first two bands by displacement of the lines in the opposite direction gave 5006·2 and 4996·1.

To test the capabilities of the new micrometer on faint spectra, two determinations were made of the wave-length of a line in the oxide-of-carbon spectrum given by a vacuum-tube enclosing coal-gas. The wave-length of this line I found before by measurement with a three-prism instrument to be 5195. It was obtained very faintly by a feeble discharge through the tube, and was measured by using as reference-line the lines given by a Bunsen flame.

Two successive determinations (each the mean of four readings) gave 5196 and 5196.

The advantages of the new form of micrometer seem to be (1) great precision in results, (2) convenience in use. My thanks are due to Mr. Browning for the skill with which he carried out my wishes in the construction of the instrument.

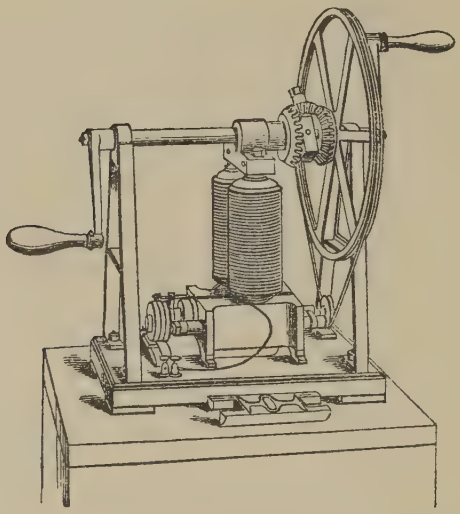
XXI. *On a new Form of Dynamo-Magneto-Electric Machine.* *By* S. C. Tisley.

In the first machines constructed by Siemens and Wheatstone in 1867 (see Royal Society's Transactions) the power of augmenting the magnetism by successive currents developed from the original residual magnetism contained in the iron, was fully demonstrated, and it was shown that the power of the machine could thereby be increased to a great extent; but the only means of obtaining external work was by the insertion in the circuit of a magnet or coil so that the secondary discharge could be utilized. Sir Charles Wheatstone also showed that a great part of the current could be shunted through a platinum wire, care being taken that the resistance of the platinum wire was sufficient to compel a large portion of the current to pass round the electromagnet. In the same year the writer designed a machine in which two separate armatures were employed, one for magnetizing the machine, the other for external work. From this machine very good results were obtained.

The machine now about to be described is a still further modification, in which the greatest amount of simplicity and effective power are combined.

The apparatus consists essentially of an electromagnet with shoes, forming a groove in which a Siemens armature is made to revolve: this is much the same as the original machines made by Siemens and Wheatstone; but there is a difference in the break or commutator, from which the currents are conducted by means of two springs or rubbers. It consists of three rings; one of these rings is complete for three quarters of the circle, the other quarter being cut away; another ring is cut away three quarters, leaving the one quarter. Between these two rings is a third ring insulated and connected with the insulated end of the wire wound round the armature; on this centre ring are projecting pieces, one a quarter of a circle and the other three quarters, so arranged as to complete the two outer circles. The rubber spring which comes in contact with the quarter of the middle ring is connected with the electromagnet of the machine, and the armature is so arranged that at the time of contact the best magnetizing current is developed. The other spring rubber is in connexion with the wire on the

armature during the other three quarters of its revolution; and this is connected with any external piece of apparatus to be worked.



By this arrangement, the alternate currents being utilized, they are all in the same direction; and owing to the length of contact the whole of the current is obtained in the best condition for heating wires, decomposing water, giving an electric light, &c.

At present only a model machine has been constructed on this principle, the armature, on which is wound about 50 feet of cotton-covered copper wire, No. 16, B. W. G., measureing 5 inches in length and 2 inches in external diameter. The magnet is formed with about 300 feet of covered copper wire, No. 14, B. W. G.; and the whole instrument, without the driving-gear, weighs 26 lbs.; with this apparatus 8 inches of platinum wire, .005, can be made red-hot, water can be rapidly decomposed, &c.

The armature is specially constructed so as to prevent the accumulation of heat, a fault to which every class of dynamo-magneto-electric machine is liable. It is made in two halves, a groove of a zigzag form being cast in each half, so that when the two are screwed together a continuous channel is maintained through the bearings for a current of cold water to pass during the whole time the machine is at work.

The advantages secured by these arrangements are extreme simplicity, and a small number of parts, only one armature and one wire being used.

The principle of utilizing the alternate currents is also applicable to machines constructed with multiple armatures; and the economy thereby resulting would prove of great advantage, as the power of the machine could be varied by throwing into the electromagnets either every other current, or every fourth, sixth, or eighth current, according to the strength required in the machine, the whole of the other currents being utilized for the electric light or otherwise.

XXII. *On Stationary Liquid Waves.*

By FREDERICK GUTHRIE.

THE following numerical results of experiments may be of use in the further study of wave-motion. They refer to stationary waves mainly, and include those of circular and those of rectangular troughs. The guiding idea was:—to deduce the velocity of wave-progression under different conditions from the frequency of recurrence of a given phase in the same place, in a manner similar to that employed in the measurement of the velocity of a sound-wave through a solid by the pitch of the note when a certain length of the solid vibrates longitudinally; and, by making use of reflection, to bring into a compact form for study the effects of wave-progression.

Circular Troughs.

§ 1. The first condition of motion examined in circular troughs is that which I shall call binodal. This motion can be easily set up in a cylindrical trough of water if it be not much less than 9 inches in diameter, nor the water much less than 6 inches deep. The bottom of an empty beaker glass serves well to generate the waves, on account of its buoyancy. This is placed on the centre of the water, and moved down and up at a rate dependent upon the diameter of the vessel. The motion of the water soon guides the hand, and, being judiciously humoured establishes itself and continues for many minutes with great regularity. The number of times in one or more minutes that

the crest appears in the middle may be easily counted if the reflection of a window-sash is seen obliquely on the liquid surface.

§ 2. *Influence of amplitude.*—Before examining the influence of diameter (which with stationary waves amounts to an examination of the influence of wave-length), a brief examination of the influence of Depth, Temperature, Density, and Amplitude had to be undertaken, to determine at all events the margin of variation admissible without vitiation of the result. The amplitude was measured by standing a brass wire graduated in millimetres, and fastened to a leaden foot, in various parts of the trough after the wave-system was established.

§ 3. A cylindrical vessel, B, of tin plate had the mean diameter 456 millims. (It was slightly elliptical, 455, 457 millims.) The depth of the water in it was 290 millims., and its temperature 16° C. Being set in binodal motion, the gauge was put in and the amplitude allowed to subside to 30 millims. at the centre.

In the 1st minute 122 recurrences were counted.
„ 2nd „ 122·5 „ „
„ 3rd „ 122 „ „

There was an interval of 10″ between the 1st and 2nd minute, and an interval of 15″ between the 2nd and 3rd minute. The amplitude of motion at the centre had now sunk to less than 1 millim. This shows that in this system amplitude or wave-height is without sensible influence. This is confirmed by the results of the following experiment, in which the initial amplitude at the centre was again 30 millims., and in which the number of pulsations made in 4′ was counted in four experiments.

(1) Amplitude at first at centre 30 millims.; in 4′ there were 490
(2) „ „ „ „ 489·5
(3) „ „ „ „ 489·5
(4) „ „ „ „ 489

or B_n mean in 1′ = 122·4.

§ 4. *Effect of Temperature.*—The water in vessel B was now heated to 41° C., or through 25° C. It had the same depth as before. The following three observations were made:—

(1) In 3′ there were 367 pulsations.
(2) „ „ 366 „
(3) In 3′ 25″ „ 420 „

which gives for 1′ the mean number 122·4, precisely the same result as for T = 16°. Temperature, therefore, has no appreciable influence, and its variation may be neglected.

§ 5. *Effect of Depth.*—For the examination of this influence the vessel B was again employed. The maximum amplitude at the centre was in all cases, at the beginning, about 30 millims.

Depth.	Time.	Number of pulsations counted.	*n*.
millim.			
290	4	489·5	122·4
270	4	490	122·5
250	3	368	122·6
230	4	490	122·5
210	3	368	122·6
190	3	367	122·3
170	3	367	122·3
150	3	366	122·0
130	3	364·5	121·5
110	3	360	120·0
90	2	234	117·0
70	2	225	112·5
50	1	102	102·0

At a depth of 30 millims. no fundamental undulations could be preserved. In order to see whether the effect of shallowness took most effect on waves of larger or smaller amplitude, the depth 70 millims. was restored and an initial central amplitude of 30 millims. was given. In the first minute 113 were counted; after a lapse of 10″ the same number 113 were counted in the second minute. A second experiment, in which the interval between the two minutes was 15″, gave for both minutes exactly the same number 113. The general conclusions to be drawn from these results appear to be that the normal rate of pulsation is not reached unless there be at least a depth of about 130 to 170 millims. (say 6 inches), and that the effect of shallowness is as great on small as on larger amplitudes. No doubt the diminution in rate is due to the increased drag upon the bottom of the vessel. It is at about this depth that the to-and-fro motion of the particles at the bottom of the vessel ceases to be sensible.

But the examination of the effect of depth is not complete unless we know whether that effect is a function of the diameter or wave-length. Accordingly a circular vessel D was taken of the diameter 314 millims., with the following result:—

Depth of water.	Number of pulsations in 1 minute.	Mean *n*.
millim.		
150	149, 149, 149	149
130	149, 149, 148	148·7
120	by interpolation	147·9
110	147, 147, 147	147
100	by interpolation	146·9
90	145, 145, 145	145·7
80	143, 143, 143	143
70	142, 142, 142	142
60	139, 139, 139	139
50	134, 134, 134	134
40	126, 125, 126	125·7

It appears, then, that in the vessel D, whose diameter is 314 millims., the effect of the shallowness begins to be felt at rather lesser depths than is the case in the wider troughs. At depths of 30 millims. and under, the fundamental system cannot maintain itself.

§ 6. *Effect of variation in chemical nature.*—A cylindrical vessel, which I shall call D, had the diameter of 321 millims. It was filled to a depth of 80 millims. with methylated spirit. This gave in each of three experiments 143 pulsations per minute. Water of the same depth gave also exactly 143. We may conclude that, as in other oscillatory motions, the density of the substance is without effect. This was shown experimentally, in the case of waves progressing in rectangular troughs, fifty years ago by the brothers Weber.

§ 7. *Effect of variation in the diameter.*—I used chiefly four cylindrical vessels. Their dimensions were as follows:—

Vessel A, mean diameter at surface of water 595 millims.
„ B, „ „ „ 456 „
„ C, „ „ „ 366 „
„ E, „ „ „ 300 „

Experiments with A (depth = 290 millims.).

In 3′ there were 321 pulsations,
„ 5′ „ 535 „
„ 4′ „ 427 „
„ 10′ „ 1071 „
or mean $A_n = 106{\cdot}96$.

Experiments with B were given in § 3, whence it appeared that (depth =290 millims.)

mean $B_n = 122{\cdot}4$.

Experiments with C (depth = 285 millims.).

In 3′ there were 411 pulsations,
„ 3′ „ 410 „
„ 3′ „ 410 „
or mean $C_n = 136$.

Experiments with E (depth = 870 millims.).

In 1′ there were 151 pulsations.
„ 1′ „ 151 „
„ 1′ „ 151 „
or mean $E_n = 151$.

Putting the results together,

	Diameter.	Number of pulsations per 1′.
A . . .	595	106·96
B . . .	456	122·4
C . . .	366	136·73
E . . .	300	151

§ 8. In the binodal system of undulations here set up it is clear that the wave-length is equal to the diameter of the cylinder. If, then, the rate of progression is directly proportional to the square root of its length, we must have $v = C\sqrt{\text{diameter}}$, where C is a constant. But the length of path from the centre to the circumference and back is the diameter; and the rate of pulsation (that is, the rate of recurrence of the same phase) must for uniform motion vary inversely with the path or diameter. Accordingly if n be the number of undulations in a time-unit,

$$n = C\frac{\sqrt{\text{diameter}}}{\text{diameter}},$$

or $n\sqrt{d} = C$.

We should get a constant on multiplying the square root of the diameter (or radius) of a circular trough with the number of pulsations per 1′. In the following Table this is done. In column 1 are shown the values of $n\sqrt{d}$. For more palpable comparison, in column 2 are the numbers got by dividing each of column 1 by the least (which is the first).

	(1) $n\sqrt{d}$.	(2)
A . . .	2608·035	1·0000
B . . .	2613·747	1·0022
C . . .	2615·799	1·0029
E . . .	2615·397	1·0029

The close coincidence of these numbers establishes the law that the rate of wave-progression varies directly with the square root of the wave-length. The absolute velocities of progression of waves of the lengths established in the troughs are—

A . . .	63·6412 metres per 1′,
B . . .	55·8144 „
C . . .	50·0437 „
E . . .	45·3000 „

and of course these numbers, divided by the square roots of their respective wave-lengths, are constant.

According to A, a wave 1 metre long travels at the rate of

83·3060 metres in a minute.

According to B, at	82·6053	„	„
„ C, „	82·7224	„	„
„ E, „	83·6691	„	„

Or taking the mean, we may conclude that a wave a metre long would travel at the rate of 83·07 metres in 1′ (a little over 3 miles an hour) if it expanded circularly, and moved freely and automatically without change of wave-length.

§ 9. *Form of fundamental binodal wave in cylindrical trough.*—The beautiful smoothness and persistence of the stationary binodal waves in cylindrical vessels enables us to examine their form with some accuracy. It is at once seen that while the only part of the water free from radial motion is that in contact with the walls and the very axis of the cylinder, there is a nodal ring of constant height—which, however, is only a geometrical expression; for the water sweeps to and fro through the node radially. Putting aside for the present the internal motion of the water, let

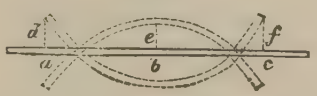
us examine the position of the nodal ring. This can be done by shifting the gauge of § 2 until there is no rise and fall on its stem, and then measuring its distance from the edge. A permanent record of the nature of the wave-system can be got by immersing a sheet of cardboard in a vertical plane passing through the axis of the cylinder and reaching to the circumference. The cardboard loses its smoothness where the water has touched it. By both of these methods it appeared that in vessel A the node was 99 millims. from the edge, in B it was 75 millims., and in C it was 64 millims. The diameters of these being respectively 595, 456, and 366 millims., the fraction of the diameters at which the nodes were formed were respectively 6·01, 6·08, and 5·72. The central amplitude was in each case 26 millims. In the vessel C this amplitude causes incipient breakers, so that the nodal point sways. It appears that in a perfect system of such waves the nodal line is $\frac{1}{6}$ of the diameter from the circumference.

§ 10. *Relative amplitude at different parts.*—When the amplitude is great, the elevation at the centre exceeds the depression at the same place. The elevation may be so great as to project water repeatedly, while the corresponding cavity is smooth and round. In such a condition the node sways. With smaller amplitudes, the alternate depression and elevation at the centre are equal to one another, and the node becomes stationary.

With great amplitudes, that at the centre is indefinitely greater than that at the edge (when the node sways). With very small amplitudes the amplitude at the edge is nearly exactly one half of that in the centre; this is the more nearly true the less the amplitude, and would, I conceive, be strictly true if the waves were conical instead of being dome-shaped.

§ 11. *Comparison with stationary waves of solids.*—A uniform elastic rod or lath, *a b c*, fig. 1, can be set in vibration in the manner shown by the dotted lines. It divides itself in such a way that $a=c=\frac{1}{2}b$ and $d=e=f$. The nodes are at $\frac{1}{4}l$ from the ends, and the amplitude at the ends is nearly the same as in the middle. A lath consisting of two isosceles triangles (fig. 2) can be thrown into similar waves whose

Fig. 1.

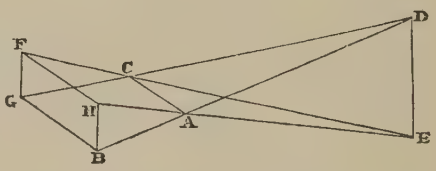


planes are at right angles to the plane of the lath (fig. 2). But the nodes are now at nearly $\frac{1}{6}l$ from the ends, and the amplitude in the middle is approximately twice that at the ends.

Fig. 2.

Imagine a thin circular board to consist entirely of such sectorial laths. They would, if strung on a circular wire passing through their centres of gravity, form a rigid system; for motion around the wire would cause gaping between the apices and crushing between the bases of the sectors. Nevertheless such motion is possible if the disk is elastic, as is shown by the formation of a nodal line around a circular plate at a distance of one sixth of the diameter from the circumference when the disk is bowed through a hole in the middle. The motion of such a disk is analogous in many respects to that of water in a circular trough. We may compare both cases with systems of isosceles triangles oscillating about lines passing through their centres of gravity. With the disk we have momentum and inertia controlled by elasticity; in the wave they are controlled by gravity. If the surfaces of the waves were conical, the following simple comparison might be drawn for an elementary displacement. Let fig. 3, D E, be the axis of the cylindrical trough. Let D G, D B be the limiting radii of the displaced cone-sector D G B. When D sinks to E, G B will be at F H.

Fig. 3.



If the sector turns on the line C A parallel to its base and one third B D from it, then D E=2F G. When D sinks to E, all the water in the tetrahedron (A C D E) leaves it, and as much water has to enter the wedge F G B A H C. These volumes must be equal. It is easy to show that they are so. This is in accordance with the general law that when two heavy planes in one plane turn together round any axis passing through their common centre of gravity, they sweep out solids of equal volume.

As the water is constrained neither to separate in the middle nor from the walls, the sectors change from conical to circular sectors and back again. And with conical waves the actual volume of water concerned in the motion is less than that traced out by the rotating sector; but this defect is the same in volume with the wedge as with the tetrahedron.

§ 12. *Level of node.*—When the wave-system is perfect, the vertical height of the node is the same as that of the surface of the liquid when at rest. The movement of particles of powder on the water shows that though the node itself is at rest, the water composing it is in violent motion. It is a stationary market in a shifting population. It shows that, as in solid and gaseous waves, a node is a region where most work is done, and accordingly where least motion ensues—a hinge.

Rectangular Troughs.

§ 13. Four troughs of zincked iron were ordered of the common depth of 1 foot, the common width of 1 foot, and the respective lengths of 1 foot, 1 foot 6 inches, 2 feet, 2 feet 6 inches. They were found to have the following dimensions in millimetres:—

	Width.	Depth.	Length.
W . . .	320	315	308
X . . .	320	315	463
Y . . .	315	315	619
Z . . .	322	315	767

About the depth, my only care was to have such a depth of water that there should be no sensible drag by friction. This appears from § 5 to be the case with about 150 millims. of water. It will be shown in § 23 that, at this width, variation in width is without sensible effect. The mean widths are given. The lengths are also the means of twelve measurements 1 inch apart. In no case was there a difference of more than 1 millim. between lengths which should have been identical.

In rectangular troughs two chief wave-systems can be established which maintain themselves automatically. They may be called binodal and mononodal respectively. The first is the counterpart of the circular waves which we have been considering. The binodal system is got by placing a wooden lath, 2 or 3 inches wide and nearly as long as the trough is wide-and provided with a handle at right angles to its plane, on the

surface of the water exactly halfway from either end, and depressing and raising it intact. The waves soon grow and maintain themselves for several minutes. The second or mononodal system is very easily produced by slightly tilting the trough repeatedly around one of its lower end edges. We shall consider the binodal motion first. The troughs were all filled to a depth of 260 millims. As it appears from the whole of the subjoined experiments that the number of undulations in the first minute was the same as those in the second, and so on, it followed that, as in circular troughs, the amplitude was without effect upon the frequency of pulsation.

§ 14. *Variation in length. Experiments with* W.

In 1′ there were 136 pulsations.
„ 1′ „ 135·5 „
„ 1′ „ 136 „

Mean number of pulsations in 1′ for W . . . 135·8.

In this square trough it is difficult to avoid the travelling of the nodal lines till they become diagonals.

Experiments with X.

In 1′ there were 111 pulsations.
„ 1′ „ 110 „
„ 1′ „ 110 „
„ 2′ „ 221 „

Mean number of pulsations in 1′ for X . . . 110·4.

Experiments with Y.

In 2′ there were 189 pulsations.
„ 2′ „ 189 „
„ 2′ „ 189 „

Mean number of pulsations in 1′ for Y . . . 94·5.

Experiments with Z.

In 2′ there were 169·5 pulsations.
„ 3′ „ 253·5 „
„ 4′ „ 338 „

Mean number of pulsations in 1′ for Z . . . 84·5.

Before collating these results we must examine the effect of shallowness.

§ 15. *Effect of depth on binodal undulations in rectangular troughs.*

Depth.	Number of undulations per 1′ in			
	Z.	Y.	X.	W.
millim.				
150	.....	.....	.....	133
170	.....	.....	.....	134
190	.....	.....	.....	135·8
210	82	92	109	135·6
220	83	93	110	135·4
240	84	94	110·4	135·6
260	84·5	94·5	110·4	135·8
280	84·5	94·5	110·4	136
300	84·5	94·5	110·4	135·6

From this Table it appears that in rectangular troughs of the same width and different lengths, a diminution in the depth of the water takes effect in increasing the period on the longer trough before it affects the shorter one. In other words, to exhibit its normal (*i. e.* maximum) rate, a short trough need not be so deep as a longer one.

It also appears that the depth 260 millims. employed in the experiments in § 14 is sufficient.

We may therefore compare the results of § 14.

	Length.	Number of pulsations.
Z . . .	767	84·5
Y . . .	619	94·5
X . . .	463	110·4
W . . .	308	135·8

The wave-length is here, of course, the length of the trough. If the rate of progression varies directly as the square root of the wave-length, then, since the rate of repetition of phase varies inversely as the path, which is also the trough-length, we should get (as in § 8) a constant on multiplying the number of pulsations in a minute with the square root of the trough-length. This is done in the following Table, where also the quotients got by dividing all by the smallest are given :—

	$n\sqrt{l}$.	
Z . . .	2340·21	1·0000
Y . . .	2341·14	1·0004
X . . .	2375·52	1·0108
W . . .	2383·29	1·0183

These numbers are even more nearly in accord than those of circular troughs of § 8. They show that the rate of progression of a restrained wave in a trough varies also directly as the square root of its length. But the question asks itself, How is it that $n\sqrt{l}$ of straight troughs is uniformly less than $n\sqrt{d}$ of circular ones? The mean value of the constant in the circular system is 2618·652; in the rectangular it is 2360·04. These are in the ratio of 1·109 to 1. To this point I shall return in § 23.

The absolute velocities of progression of the waves are:—

Z	. . .	64·8115 metres in 1′.
Y	. . .	58·4955 „
X	. . .	51·1152 „
W	. . .	41·8264 „

If we assume the formula

$$v_1 = v_2\sqrt{\frac{l_1}{l_2}}$$

to be true for the wave-length of a metre (and we see it is approximately true for 767 millims.), it is found that a wave a metre long in a rectangular trough travels,

According to	Z,	at the rate of	74·006 metres in 1′.
„	Y,	„	74·348 „
„	X,	„	75·088 „
„	W	„	75·371 „

The mean of these is 74·7, and this is the distance which a wave a metre long would pass over in 1′ when moving in deep water between parallel walls. This is under the assumption that no velocity is lost by reflection; and the same assumption must be made in § 8 with circular troughs, where the velocity of the metre wave in circular troughs was found to be 83·1 metres in 1′.

§ 16. As the circular undulation was referred to an oscillating triangle or vibrating pair of triangular laths, so may the motion of the water above described in the rectangular trough be likened to the balancing of two rectangular laths supported at their lines of gravity, or, more closely, to the binodal vibration of a rectangular lath (see § 11). The lines of gravity are now from each end $\frac{1}{4}$ of the joint length. The nodes are at $\frac{1}{4}l$ from each end. Accordingly in rectangular troughs the nodes are at exactly $\frac{1}{4}l$ from the ends when the undulation is binodal. Also, on taking a paper section of the system (see § 9), it is found that the am-

plitude at the centre only exceeds that at the ends by a very small amount. As before, the node is not a line of material rest. The water sweeps through it towards the mountain which is forming at one side or the other.

§ 17. *Mononodal undulation in circular troughs.*—In circular troughs the mononodal waves can be produced by tilting or by a segmental stirrer. The former, of course, lifts the node and produces at first a longer wave than is due to the proper limits of the trough. This very fact causes the wave to accommodate itself to the trough, and to establish a nodal line which is a diameter of the cylinder at right angles to the plane of the wave. The same system is produced at once by the careful use of the segmental stirrer. Two kinds of irregularities may show themselves. The one depends upon the partial degeneration of the mononodal system into circular bi- or quadrinodal systems; the second consists of an almost unavoidable rotation of the node. Such rotation, which need not exceed a quadrant in three or four hundred waves, must of course be an experimental fault, since no reason can be assigned why it should take place in one rather than in the other direction. Neither irregularity appears to affect the rate of the mononodal system; the first gradually effaces itself probably by disaccord between the older and newer effects.

§ 18. *Effect of depth on the mononodal system in circular troughs.*

Depth.	Number of undulations per 1′ in	
	A.	B.
millims.		
145	63	76
155	.....	78
165	.....	79
175	65·5	80
195	.....	81
225	.....	82
255	71	83
280	71·5	83·5
330	71·6	83·5

§ 19. *Effect of variation in diameter on the above experiments with* A (depth = 290 millims.).

In 2′ there were 143 waves.
„ 2′ „ 143 „
„ 2′ „ 143 „
Mean rate for A per 1′ . . 71·5.

The second and third 2′ were consecutive, and accordingly the wave-rate of sequence is independent of amplitude.

Experiments with B (depth = 290 millims.).
In 2′ there were 167 waves.
„ 2′ „ 167 „
Mean rate for B per 1′ . . 83·5

Experiments with C (depth = 185 millims.).
In 2′ there were 183 waves.
„ 2′ „ 183 „
Mean rate for C per 1′ . . 91·5.

Experiments with E (depth = 870 millims.).
In 1′ there were 103·5 waves.
„ 1′ „ 103·5 „
„ 1′ „ 103·5 „
Mean rate for E per 1′ . . 103·5.

Putting the results together—

	Diameter.	Mean n.	$n\sqrt{d}$.	Depth.
				millim.
A	595	71·5	1739·09	290
A′.....	595	71·6	1741·53	330
B	456	83·5	1782·06	290
C	366	91·5	1760·49	185
E	300	103·5	1789·62	870

It appears both from § 18 and from the above, that A or even A′ was scarcely deep enough to give the normal rate for their great wave-lengths. It also appears that for the shorter system of C the depth 185 is insufficient. From § 18 it seems that the maximum or normal rate is reached. We may take the nearly identical numbers for B and E for the valuation of the mean constant, which is 1785·84 (in metre-minute measurement).

§ 20. *Mononodal undulations in rectangular troughs.*—In rectangular troughs the mononodal wave-system can be established by tilting or by the use of a wide stirrer. The wave-length is twice

that of the trough. As the shallowness influences longer waves more than shorter ones, we must look for greater discrepancies here than in §§ 15, 16, 17. Taking the same troughs, namely Z, Y, X, W, the first experiments were to determine the influence of depth.

Influence of Depth on Mononodal Waves in Rectangular Troughs.

Depth.	Z.	Y.	X.	W.
millims.				
120	.....	.....	.....	92
140	.....	.....	.....	93
160	.....	.....	.....	94
180	.....	.....	.....	94
200	49	.....	.....	94
220	51·5	.....	.....	94
240	53	61	74·6	94
260	54	62·2	75·2	94
280	55	63	76	94
300	55	63	76	94

Experiments with Z (depth =290 millims.).

In 2′ there were 110 waves.
„ 3 „ 165 „
Mean rate for Z per 1′ . . . 55

Experiments with Y.

In 3′ there were 189 waves.
„ 2′ „ 126 „
Mean rate for Y per 1′ . . . 63

Experiments with X.

In 3′ there were 228 waves.
„ 2′ „ 152 „
Mean rate for X per 1′ . . . 76

Experiments with W.

In 1′ there were 94 waves.
„ 1′ „ 94 „
„ 1′ „ 94 „
Mean rate for W per 1′ . . . 94

Collecting and multiplying each number of waves by the square root of its trough-length (trough-length = $\frac{1}{2}$ wave-length, the ratio is the same), and dividing all such products by the least of them, we get the following numbers :—

	Length of trough. l.	Mean number of waves in 1′ $=n$.	$n\sqrt{l}$.	
Z . . .	767	55	1523·21	1·00
Y . . .	619	63	1568·42	1·03
X . .	463	76	1635·32	1·07
W . .	308	94	1649·70	1·08

In spite, therefore, of the apparent attainment of maximum rate at 280 millims., we must reject the two longest waves in adopting a constant, which, taking the mean of the two shorter systems, is 1642·5.

In all the four cases, namely mono- and binodal waves in circular and rectangular troughs, the value of $n\sqrt{l}$ or $n\sqrt{d}$ increases as the length of l or d decreases. The increase of rate of wave progress due to increased wave-length is somewhat more than counterbalanced by the increased path, so that the "constant" is larger with smaller troughs.

§ 21. *Comparison between mononodal and binodal waves in rectangular troughs.*—In the same trough the wave-length of the mononodal system is always that of the binodal system, the former being $2l$ ($l=$ trough-length). If the ratio of progression be as the square root of the wave-length, then the number n of waves in a given time, varying directly as the rate of progression and inversely as the distance, in the same trough

$$\frac{\text{number of binodal}}{\text{number of mononodal}} = \sqrt{2}.$$

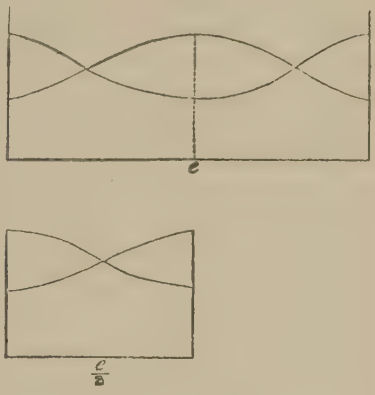
How far this is borne out in the experiments is seen in the following comparison of the shorter troughs, where the mononodal system is normal.

	Length of trough.	$\sqrt{2}\times$ mononodal.	No. of binodal.
X . .	463	107·47	110·4
W . .	308	132·94	135·8

§ 22. If we take a binodal system and insert a rigid plane diaphragm vertically down the middle and at right angles to the plane of the wave, we divide it under little disturbance obviously into two mononodal systems of half the trough-length, but of the same wave-length as before. These should oscillate (inde-

pendently of one another) at the same rate and at the rate of the original system. The experiment which illustrates this relation most directly is Y(l=619 millims.) of the binodal, where n=94·5, and W (l=308) of the mononodal, where n=94·0. Such a material diaphragm cannot satisfy the ideal conditions; for if at rest, additional friction is introduced by the motion along its surfaces of pairs of particles at each side exerting equal and opposite pressures. These, in the absence of the diaphragm, exert no friction on one another, since they move at the same rate.

Fig. 4.



§ 23. *General comparison with Pendula.*—As the amplitude of the wave produces no sensible effect on its rate of progression, and the latter varies for small wave-lengths directly as the square root of the wave-length both in circular and rectangular troughs, the oscillation of water may be closely compared with that of a pendulum, which for small circular arcs is isochronous, whatever be the amplitude of displacement; height of wave : height of pendulum-bob $= v(1 - \cos \frac{1}{2}\theta)$. The mean velocity of the pendulum-bob varies as the arc for the same radius; for the same small angular displacement inversely as the square root of the radius. We have then the analogous cases:—

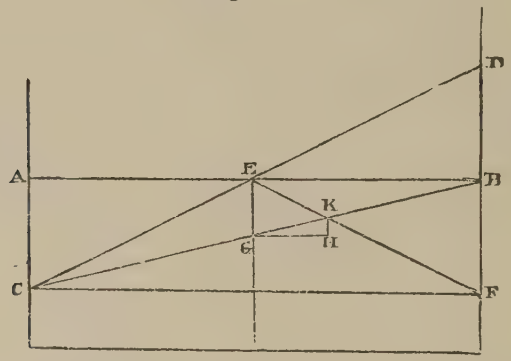
Pendulum.	Water-wave.
$n \sim \frac{\text{velocity}}{\text{path}}$,	$n \sim \frac{\text{velocity}}{\text{path}}$,
$\sim \frac{\sqrt{r}}{r}$,	$\sim \frac{\sqrt{l}}{l}$,
$\sim \frac{1}{\sqrt{r}}$,	$\sim \frac{1}{\sqrt{l}}$.

The moving force in a simple pendulum is always equal to its weight, though this force is always applied partly to the point of suspension. In the moving water the to-and-fro motion is

due to the action of a variable difference of pressures (heights). The law of oscillation, however, is preserved, although the undulations are the result of particle-motion in the mass of the liquid, and although these particles in the case of stationary waves, as was pointed out so many years ago by the brothers Weber, do not complete their orbital motion, but themselves swing, like the bobs of elementary pendula, to and fro in short-arc'd rhythmic sequence. Whenever the momentum and inertia vary together (as in all cases of solid magnification—*i. e.* alteration of size, conservation of shape, and either conservation or alteration of density), the pendular law must be preserved, and this whether the mass to be moved is wholly active, as in the pendulum itself, or more or less passive, as the logan oscillation of a common balance.

In rectangular troughs it is easily shown that, if the surface is plane, the pendulum (elasticity) or torsion law must hold good, and that the force of restoration is proportional to the linear measure of disturbance. If A B (fig. 5) be the level of

Fig. 5.



the water in a rectangular trough and it is set in mononodal undulation, so that in one phase the surface is at C E D, the line B D, B F, or A C is half the amplitude. Consider all the water below C F to be without influence. The rectangle of water A C B F has become the triangle C F D. Join C B and E F; their intersection K is the centre of gravity of C D F, and is $\frac{2}{3}$ of C B from C, &c. Draw G H horizontal and K H vertical till they intersect. The centre of gravity of the rectangle A C B F has been raised by the change of form from G to K—

that is, from H to K. The work in the displaced water is $DCF \times KH$.

$$KH = KG\frac{BF}{BC},$$

and

$$KG = (\tfrac{2}{3} - \tfrac{1}{2})BC = \tfrac{1}{6}BC,$$

or

$$KH = \tfrac{1}{12}DF.$$

The work done and potential at maximum excursion vary directly with displacement. What is true for a plane is of course true for a rectangular trough. It can also be shown to be true for disturbances in circular troughs from fig. 3, § 11, and follows indeed without further proof from the equal tetrahedron and wedge having been generated by the rotation of a triangle balanced about one of its lines of gravity. The fact that the surface of both wave-systems is curved, and that this curve involves the cohesion and viscosity of the liquid, puts the exact solution of the question beyond the present power of analysis.

§ 24. *Absolute comparison of circular waves with pendulum.*—It appears that both the binodal waves in circular troughs and the binodal waves in rectangular troughs follow the pendular law, and that the mononodal waves in rectangular and circular troughs do so nearly. It follows from § 23, where it is shown that no dead weight has to be stirred, that, if the liquid has no viscosity, not only should the pendular or torsion law hold good, but the actual rate of undulation in the binodal system in the unhampered or circular wave should be the same as that of a pendulum of radius equal to half the wave-length—that is, to the diameter of the cylinder. If a pendulum of length l be suspended over a circular trough of radius l, it keeps time very nearly with the fundamental undulations in the trough. This can be beautifully shown in an experiment where the pendulum-length can be altered till it is isochronous with the pulsations. But it appears best from the data already given. Taking the length of the seconds-pendulum at London at 994 millims., the number of oscillations which pendula of the radii of the circular troughs give is shown in the following Table,

$$n' = 60\sqrt{\frac{994}{r}}.$$

	(1) Radius of trough, r.	Number of liquid undulations.	(n'). Number of oscillations of pendulum of length r.
A	297·5	106·9	108·6
B	228	122·4	125·3
C	183	136·7	139·4
D	157	149·0	150·9
E	150	151	154·4

The pendulum-oscillations exceed the liquid ones, on an average, 2·8; on the corrected average, under the omission of E, 2·05. The mean per-cent. deficit is about 1·5. Considering that the liquid undulations do not, even in the largest troughs, sustain themselves for more than 20′, while a pendulum in air will oscillate for hours, and remembering that rate of decadence is nearly proportional to individual delay, I do not think that the above numbers are at all in disaccord with the assertion that the *fundamental or binodal circular undulations in an infinitely deep circular vessel are isochronous with those of a pendulum whose length is equal to the radius of the vessel.*

In hemispherical and cycloidal vessels I anticipate that a still nearer identity would be obtained. The exact form of the cup of revolution which would give the nearest approach to the pendulum-rate depends of course upon the shape of the wave; and here we are again referred to the complex influence of viscosity and cohesion.

§ 25. *Comparison of rectangular troughs with pendulum.*—Making a similar comparison to the above with rectangular troughs having binodal undulations, let us find what are the lengths of pendula isochronous with the water-waves in the troughs Z, Y, X, W respectively. The number of liquid undulations being n, the formula

$$l = \left(\frac{60}{n}\right)^2 994$$

gives us the required lengths of the pendula in millims.

	Trough- or wave-length.	Number of liquid undulations.	Length of isochronous pendulum.	$\frac{2}{\pi}\times$trough-length.
Z	767	84·5	501·194	488·4
Y	619	94·5	400·482	394·1
X	463	110·4	293·627	294·8
W.....	308	135·8	194·028	196·1

It appears, therefore, that *the binodal undulations in a rectangular trough are isochronous with the oscillations of a pendulum whose length is* $\frac{2}{\pi}$ *of the trough or wave-length.*

§ 26. *Comparison between circular and rectangular troughs in binodal vibration.*—Briefly, it has appeared that the differences between the wave-systems in circular and rectangular troughs, both being binodal, are as follows:—In rectangular troughs the nodes are one fourth of the trough-length from the ends. In circular troughs they are one sixth of the diameter. In rectangular troughs the amplitude is a very little more at the centre than at the edges. In circular troughs it is very little more than double that at the edges. The rate of progression in both varies as the square root of the wave-length; and the rate of sequence, or n, varies inversely as the square root of the wave-length. The waves of the circular system are isochronous with a pendulum of half the wave-length; in the rectangular trough this pendulum must be $\frac{2}{\pi} \times$ wave-length. If reflection does not alter velocity, the circular wave 1 metre long will travel 83·1 metres in one minute if it preserves its wave-length; while a wave 1 metre long between walls will only travel 74·7 metres in one minute. In answering the question as to whence this latter difference arises, a few preliminary experiments are useful. That waves in circular troughs endure longer than waves of the same amplitude in rectangular ones may be attributed in great measure to friction. That element of friction which is due to adhesion and cohesion can be directly compared as follows. In a square trough in one complete undulation the surface rubbed by the two end lines of particles is $4al$ if a is the amplitude and l the trough or wave-length. The surface rubbed by the two side-lines of particles is $2al$. The whole surface rubbed by the four edges of the top layer of particles is $6al$. In the circular trough whose diameter is equal to l, the amplitude at the centre being a, the surface rubbed by the edge ring of particles in a complete undulation is $2a\pi\frac{1}{2}l$ or $a\pi l$; or the cohesive friction on the square trough is for the upper layer 1·9 times that of the circular trough for the same wave-length and edge-wave amplitude. If such friction were the cause of the slower rate of wave-

progress in rectangular troughs, we should expect the effect to be less in wider troughs than in narrower ones, because, while the surface scraped would be the same, the mass of the water would be greater.

Accordingly the rectangular troughs W, X, Z, which have the dimensions

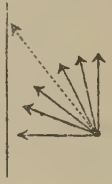
	Width.	Length.
W . . .	320	308
X . . .	320	463
Z . . .	322	767,

were filled to a depth of 280 millims., and by means of long laths set in binodal undulation in the direction of their widths. The wave-lengths of W and X were accordingly identical; and the wave-length of Z was only 4 millims. longer. The mean of four concordant determinations in each case showed in 1′,

for W . . .	134
„ X . . .	134
„ Z . . .	133·5

In brief, the length of the trough *across* which the system was established produced no sensible variation in rate of phase-recurrence, and therefore presumably none in wave-progression.

Fig. 6.



But a wave in a rectangular trough undergoes a constraint quite other than that of friction. Its motion is repressed by the sides. It presses equally and radially in all directions. The wall offers resistance perpendicular to itself. The resultant of the quadrant of the forces in fig. 6 makes an angle of 45° with the wall. The lateral pressure on the wall is $\frac{1}{\sqrt{2}}$ of the total radial pressure; the onward pressure is also equal to $\frac{1}{\sqrt{2}}$, the total pressure. The constant obstruction thus offered by the walls is the cause of the preservation of amplitude. The constant obstruction thus offered by the rigid parallel walls, while preserving amplitude, is, I imagine, the main cause of the lesser rate of wave-progress in rectangular troughs; and could exact measurement be effected, I should expect that the ratio between the times of subsidence from one amplitude to a lower one

in a circular and rectangular trough would be found to be nearly the ratio of 83·1 to 74·7. The measurements of rate of subsidence cannot be made of sufficient exactness to test this experimentally; but it is clear enough that binodal waves in circular troughs are far more enduring than those of rectangular ones.

§ 27. It may be shown that in rectangular troughs, when mononodal undulation takes place, the path of the centre of gravity is a parabolic arc, supposing the surface to remain flat. But as this condition never obtains in reality, and as no one particle remains in its motion the centre of gravity of the moving mass, this parabolic motion of the centre of gravity is of no direct account.

§ 28. *Cross binodal undulations in circular troughs.*—On pressing simultaneously two opposite points of the rims of A and B, they yield sufficiently to start undulations which are analogous to the ring of a bell or the fundamental quadrant vibration of a Chladni plate. By repeating the push at proper intervals, a well-shaped system of undulation is established, consisting of quadrant waves separated by diametral nodes at right angles to one another. The rate of recurrence of these in troughs A and B was examined. The troughs were filled with 300 millims. of water.

	Number of recurrences of same phase in 1′.
A.	94
	94
	94
B.	109
	109
	109

Taking the troughs A and B, we have the following comparison :—

Fig. 7.

A....	106·9	71·6	94
B....	122·4	83·5	109

§ 29. *Compound motions.*—The motion given to a glass of water when the axis is made to describe a narrow cylinder and then brought to rest, whereby a wave careers round the vessel, is a modification of the mononodal. If the node of such a system performs one complete revolution during the phase-life of the mononodal wave, the crest reappears in the same place. In the trough B, which held 300 millims. of water, in each of three experiments the crest reappeared in the same place eighty-four times in one minute. The travelling of the node, therefore, makes no difference in the wave-rate. The motion of the surface resembles that of a disk which is at right angles to an axis when the axis describes a circular cone.

If the water revolves on its axis in the trough, mononodal undulations are carried round in a similar manner.

The revolution of the water does not interfere with the period of the binodal circular waves. Thus the water in A, being 300 millims. deep, was stirred round; and while in rapid rotation, it was impressed with the circular binodal wave-system. In each of two minutes 107 waves were counted; and the last minute was immediately following the first. At the end of the expe-periment the water was still circulating at the rate of 12° per 1″.

§ 30. *Oil on water.*—The restraining effect of a layer of oil on the water, inasmuch as it causes the more rapid loss of amplitude, might be supposed to effect a retardation in the wave's progress. Such is not the case, apparently; for a layer of olive-oil 5 millims. in thickness undulated with the water beneath it

in the trough B at such a rate as to give in three experiments the numbers 123, 122·5, 122·5.

§ 31. *Viscosity.*—Glue added in varying quantity up to 1 lb. of the solid in the trough B did not change the rate from 123.

§ 32. *Images on wall.*—It can readily be imagined that the sun shining on water or, better, water covered with oil in one of the states of motion above described, gives most elegant figures on the wall. The circular binodal, for instance, shows lateral and vertical loop-exchange; the cross binodal circular shows a collapsing and expanding parallelogram; the binodal rectangular exhibits bars of light which approach and recede with a pause and slight regression. From the distribution of light in the images, the kind and, to some extent, the degree of curvature can be ascertained. Thus, when the circular binodal has nearly subsided, the only alteration visible is the change of luminosity in the centre, due to the alternating convexity and concavity.

§ 33. *Recent contributions to the subject.*—On the occasion of the communication of my experiments to the Physical Society, I invited the Members to offer suggestions regarding the *rationale* of some of the stationary waves which were there exhibited.

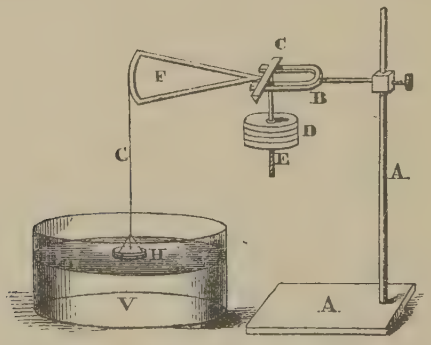
Mr. O. Lodge has been good enough to point out how completely the phenomenon depends upon the relationship between wave-length and rate of wave-progress. I regard the results as being of value, in a great measure because they show, I believe, more conclusively than can be shown by the wave-progress on open sheets of water, that this law, that the velocity of progress is directly proportional to the square root of the wave-length, is a true one. This they do by reason of the conservation of wave-length.

In 'Nature' of July 28th Lord Rayleigh discusses the theoretical aspect of the question, and shows that the wave-rate of sequence which I had given for circular binodal waves corresponds closely with the theoretical number deduced from the hypothesis that in troughs of infinite depths the variation of the diameter is the only thing which affects the wave-rate or period. That the pendulum-, or, as I should prefer to call it, the Logan-law, or law of magnification, should be so closely preserved through the complex molecular movement which actually takes place is of the greatest interest—the more so as the paths

of the particles are not increased by an increase of the trough, but there is only a diminution of the eccentricity of their fragmentary orbits. This has, of course, its counterpart in vibrating elastic rods. The mathematical interpretation of the experimental fact is very valuable. I do not suppose that such interpretation can be complete until account is taken of the cohesion and viscosity of liquids, to which, on the other hand, such experiments may supply a new weapon of attack. The other kinds of wave-systems, the rate of which Lord Rayleigh predicts mathematically, and which in rectangular troughs were described, but not investigated with regard to their period, in the elaborate work of the brothers Weber, I had already examined in regard to their period both in circular and rectangular troughs. In this respect it is safe to predict that every system of vibration of a square or round Chladni plate may be reproduced in a square or round vessel of water, and that the time-ratio will be preserved. Further, in water-wave systems we can have mononodal vibration, which for automatic motion is inadmissible in vibrating plates. With the exception of those above studied, the more complex systems appear short-lived.

34. That the wave-motion of the binodal circular system is not only isochronous with the radius pendulum, but also coincides with pendular motion throughout the whole course of both, appears from the fact that the two sustain one another when in direct integral mechanical connexion. Fig. 8 shows such an arrangement. A tuning-fork, B, having the plane of

Fig. 8.



its prongs horizontal, is fastened to a heavy stand, A. Resting

on the fork is a knife-edge, C. To C is attached a vertical screw-rod, E, carrying a nut (invisible in the figure) which supports the heavy weight D, consisting of a series of leaden disks. Also fastened to C is a cardboard sector, F. The circular edge of F is split, and into the crack is pasted a paper gutte. A silk thread, G, is fastened to F, and carries a little paraffin disk, H. The pendulum-length is altered by the nut E until the disk H is isochronous in its motion with the water in V. The vessel V is shown out of proportion small. When the right length of E is obtained, the disk H moves up and down with the water without a ripple for twenty minutes or more, showing how truly pendular is the wave-motion. It follows that, for small disturbances, the rate of vertical motion at the normal level must be closely proportional to the altitude of the wave or amplitude of undulation.

XXIII. *On the Flow of Electricity in a uniform plane conducting Surface.*—Part II. *By* Professor G. Carey Foster, *F.R.S., and* Oliver J. Lodge, *B.Sc.**

43. In an unlimited uniform conducting surface, at given points of which there are sources and sinks of electricity, the electrical condition of each point within a limited area must be determined by the electrical condition of its boundaries. Moreover, if an equal similar area be taken in a *limited* conducting sheet, whose conductivity and thickness outside this area vary in any manner, the electrical state within the area will still be the same as in the previous case, if only the conditions at the boundaries remain unchanged; for evidently the electrical flow within the area will depend on the actual electrical state of the boundaries, and not upon the way in which this state has been produced. Consequently the truth or falsehood of conclusions arrived at, like those given in Part I. of this communication, on the assumption of an unlimited uniform conducting surface, may be tested by experiments on a conducting sheet of finite size, if we can ensure that, at the edges, the electrical conditions are the same as they would be at the boundaries of a portion of the same

* The substance of this paper was read to the Physical Society on the 27th of February, 1875; but it contains also experimental results obtained since that date.

size and shape taken in an infinite sheet in which there is a known distribution of sources and sinks.

44. In order to carry out such experiments, it is needful that the boundaries of the part of the conducting sheet examined should be formed either wholly by equipotential lines, or partly by equipotential lines and partly by lines of flow. It is evident that an area within which there is a flow of electricity (unless the circuit is complete within this area, as in a conducting plate moving in a magnetic field) cannot be bounded entirely by lines of flow; for in that case no electricity could enter or leave the area.

Any line in a conducting sheet can be practically made to be an equipotential line by laying along it, and in close contact with the sheet, a conductor whose resistance in directions parallel to the sheet is insensible—for instance, a thick band of copper. And any line along which the sheet is bounded by non-conductors (*e. g.* the free edges of the sheet) is necessarily a line of flow, since no electricity can pass across such a boundary in either direction. The experimental conditions can thus be chosen so that a conducting area may be bounded either by equipotential lines wholly, or partly by equipotential lines and partly by lines of flow, placed so as to correspond to any given system of poles.

45. Methods of testing experimentally the conclusions deduced from theory, founded on the general principles above indicated, have been employed by various investigators. Thus Kirchhoff, in his paper of 1845 (Pogg. *Ann.* vol. lxiv.), traced the form and distribution of the equipotential lines on thin sheets of metal bounded by four circular arcs, two of which, being free edges, were lines of flow and belonged to the same circle; while the others, of much smaller extent than the first two, were equipotential lines determined by the contact of small cylindrical electrodes (conducting wires) with the margin of the sheet. He traced the course of the equipotential lines by fixing a wire connected with one terminal of a galvanometer in succession at several points arbitrarily chosen, and in each case finding, by trial with a wire from the other terminal, a sufficient number of points having the same potential as the selected point. To ascertain the distribution required to make the difference of potential between consecutive lines constant, he determined a number of points along the straight line joining the electrodes, such that the difference of potential between each one and the next was just

sufficient to compensate the constant electromotive force of a thermoelectric element. The form of the equipotential lines in some rather more complicated cases was determined in the same manner by Quincke*, and in other cases again by students in the Edinburgh University Laboratory†. Schwedoff‡ also has investigated the direction of the flow at various points in a rectangular sheet of tinfoil.

Recently an elaborate series of experiments, by a method similar to Kirchhoff's, has been carried out by Professor W. G. Adams§, who has also extended the same method of inquiry to conductors of three dimensions by the employment of conducting liquids (mercury, sulphate of copper, sulphate of zinc).

Verifications by Kirchhoff and by Mach of the theoretical expressions for the strength of the current at different parts of a conducting sheet have already been referred to (see Part I. § 15, footnote.

Experimental measurements of resistance, in addition to the attempts of Kirchhoff already mentioned (Part I. § 1), have been made by Gaugain||, who measured the resistance of a solution of sulphate of copper contained between two circular copper cylinders placed one inside the other with their axes parallel, and thus verified Blavier's formula (see Part I. § 24, footnote)—by v. Obermayr¶, who verified Kirchhoff's formula for circular disks, and also a formula deduced by Stefan for the resistance of an oblong strip with circular electrodes placed on the central line, by experiments with disks and strips of platinum-foil—and by Domalip**, who measured the resistance of thin circular layers of a solution of sulphate of zinc contained between parallel glass disks, the electrodes being zinc rods inserted through holes in the upper piece of glass.

* Pogg. *Ann.* (1856) vol. xcvii. p. 382.

† W. Robertson Smith, Proc. Edinb. Roy. Soc. 1869–70, p. 96.

‡ Schwedoff (Pogg. *Ann.* 1873, Ergänzungsb. vi. p. 85) inferred the direction of the resultant flow at any point of a conducting sheet in which there are two equal opposite poles, from simple geometrical considerations, to a great extent similar to those employed by Professor W. R. Smith and by ourselves (Phil. Mag. vol. xlix. p. 394). We should have referred to his paper before, if we had known of its existence.

§ Proc. Roy. Soc. (1875) vol. xxiii. p. 280.

|| *Ann. de Chim. et de Phys.* [3] vol. lxiv. p. 200 (1862).

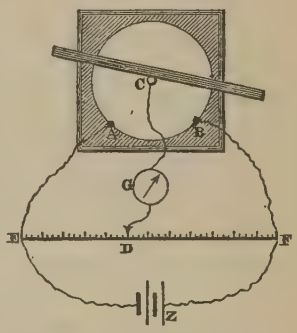
¶ *Wien. Akad. Ber.* (1869) vol. lx. pt. 2. p. 245.

** Carl's *Repertorium für Experimental-Physik*, 1874, vol. x. p. 23.

46. The results of all these investigations are in such close agreement with calculation that it cannot be considered that any further confirmation of the general theory is needed. It nevertheless seems worth while to point out, and to illustrate by specimens of the results obtained by it, a simple method which serves both for determining the form and distribution of the equipotential lines on a conductor of two dimensions, and for measuring the resistance of the part of it contained between any two given lines. In all except the very simplest cases, it would be much easier to determine the equipotential lines and resistance experimentally in this way than by the corresponding calculation.

47. The method referred to is an application of the principle of Wheatstone's bridge, and will be intelligible from fig. 1. The poles of a battery (Z) are connected with the two ends of a graduated German-silver wire, E F, and also with two fixed points, A and B, of the conducting disk on which the equipotential lines are to be traced; a galvanometer, G, is connected to a sliding contact block D on the wire E F, and to a contact-pin C, with the point of which the sheet can be explored. Under these circumstances the points C and D will be at the same potential whenever the galvanometer shows no deflection; hence if the pin C be moved about on the disk, and a mark made at every position for which the galvanometer is not affected, all these marks will lie on one equipotential line. By repeating this operation, after shifting the slider D along the graduated wire, another equipotential line can be marked out on the sheet; and any number of lines can be traced, so as to correspond to equal differences of potential, by shifting the slider through successive equal distances.

Fig. 1.



48. In the actual experiments the connexions were not made exactly as shown in fig. 1, but the positions of battery and galvanometer were interchanged. This arrangement made it unnecessary to use a separate key for closing the battery-circuit, as the circuit was completed by the contact of the pin C with the

sheet; and since these contacts were only momentary, no sensible disturbance can have arisen from unequal heating. The battery used was a single cell of Smee; the galvanometer was a reflecting one of low resistance. The disk was a circular piece of tinfoil, of 27·8 centims. diameter, pasted on the bottom of a tray or shallow box of hard wood, on the edges of which rested a wooden bar supporting the contact-pin C. This last slid in a vertical tube fixed to the wooden bar; a spiral spring inside the tube kept the pin out of contact with the sheet except when it was pressed down. When the right position on the sheet had been found by means of the blunt point of the pin, this position was marked by pressing the pin down harder, so as to make a slight indentation in the tinfoil.

The mode of making contact at A and B which we found to answer best is shown in fig. 2. A S is a brass spring with a projecting pin A, and a screw S with nut and washer. The screw passes through the wood of the tray and is fixed by screwing up the nut, the conducting wire being clamped between the nut and washer. The rounded end of the pin A rests on the tinfoil surface, pressing it firmly and making a good circular contact. It was found necessary to lubricate the bar carrying C with blacklead to prevent its causing vibrations when sliding on the edges of the tray, and so deranging the contact of the springs A and B.

Fig. 2.

49. After a series of points have been marked on the tinfoil in the manner described, a line may be drawn through each set by hand. A set of lines obtained in this way for the case of two equal opposite poles on the edge of a circular disk correspond to a portion of the diagram in Plate IV. bounded by a complete line of flow. P being any one of the dots marking such a line, if they are circles with the poles as inverse points, the ratio $\frac{AP}{BP}$ will be constant for any one line (Part I. § 16); and if they are at equal differences of potential, the values of this ratio will vary from one line to the next by a constant factor μ (Part I. § 18). The numbers given in the following Table show to what extent these requirements are fulfilled by the ex-

perimental lines. Nine or ten points on an average were determined for each line; but as a specimen of the results it seems sufficient to give the ratios found for four points only, viz. (in the case of each line) for a point near either end, a point on or near the straight line joining the poles, and a point about halfway between this and the more distant end of the equipotential line.

Values of $\frac{AP}{BP}$ $(=\mu^n)$ *for four different points on each of eight different equipotential lines.*

	μ^{-3}.	μ^{-2}.	μ^{-1}.	μ^0.	μ.	μ^2.	μ^3.	μ^4.
	·200	·335	·568	·996	1·71	2·94	5·05	8·75
	·200	·327	·568	1·00	1·70	2 95	5·13	9·01
	·190	·323	·573	1·00	1·71	2·98	5·18	8·97
	·196	·333	·573	1·02	1·73	2·98	5·53	8·91
Means.	·1965	·3295	·5705	1·005	1·71	2·96	5·22	8·66.
	$=(1\cdot72)^{-3}$	$=(1\cdot74)^{-2}$	$=(1\cdot76)^{-1}$		$=(1\cdot71)$	$=(1\cdot72)^2$	$=(1\cdot73)^3$	$=(1\cdot\ \ 7)$

In the experiments by which the above values were obtained the distance between the poles was 19·45 centims. Taking this as the value of $2a$, and 1·724 for μ, a system of coaxal circles was drawn by the method of § 21, Part I. A tracing of the experimental curves was then superposed on these theoretical circles; and the two sets were found to coincide almost exactly.

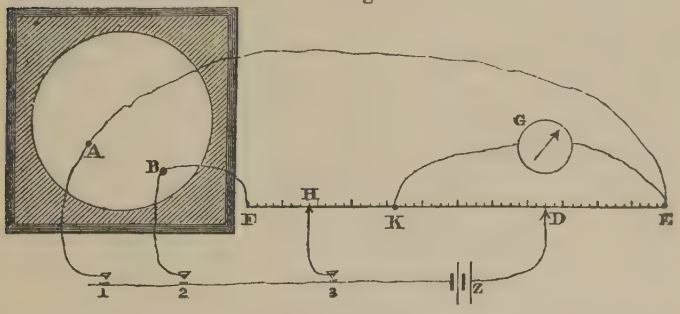
Another case tried was that of a circular disk with four equal poles on its edge, two positive and two negative; the equipotential lines obtained experimentally agreed in all particulars with those drawn by the superposition method of § 35.

50. The arrangement of fig. 1 serves not only to lay out the equipotential lines on a disk, but also to determine its resistance; for if the pin C be kept in contact with any point of the disk (with the electrode A for instance), while D is adjusted so that the galvanometer stands at zero, and if C is then moved to another point (say the electrode B) and D again adjusted, the resistance of the part of the disk between the equipotential lines which respectively pass through the two positions of C is the same fraction of the resistance of the branch E A B F, as the resistance of the part of the wire between the two positions of D is of the resistance of the whole wire E F. Accordingly the resistance of the disk is given, in terms of that of unit length of the wire E F,

as soon as the ratio of the resistances E A B F and E D F is known. In order to determine this ratio, the connexions shown in fig. 1 were somewhat modified, the arrangement finally adopted being that shown in fig. 3.

One pole of the battery is connected by means of a slider D with some point of the wire E F, while the other can be connected at pleasure, on the one hand, with either of the electrodes A and B upon the disk by pressing one of the contact-keys 1 and 2, or, on the other hand, with another slider H on the wire E F by pressing the key 3. The galvanometer is permanently connected with one end E of the wire, and with a fixed point K between the sliders D and H. The part of the graduated wire between the galvanometer-terminals E and K thus takes the place of the whole length E F in the arrangement shown in fig. 1. To measure the resistance between A and B in terms of that of the graduated wire, it is then only necessary to

Fig. 3.



press successively the keys 1 and 2, and each time to adjust the slider D so as to give no deflection on the galvanometer, and then to connect the battery with H, by the key 3, and to find the positions of D required to keep the galvanometer at zero for two positions of H separated by the known amount h. If we denote by s the distance through which D has to be shifted on the wire E K when the battery-contact on the disk is transferred from A to B, and by d the shift of D corresponding to the shift of H over the distance h, we have

$$\frac{\text{resistance between A and B}}{\text{resistance of } s} = \frac{\text{resistance of } h}{\text{resistance of } d};$$

and since s, h, and d are all of them lengths measured on the same wire, their resistances are simply as their lengths. Hence, taking as unit of resistance the resistance of unit length of the graduated wire, we have $\text{resistance A to B} = s\frac{h}{d}$.

It will be observed that the ratio $\frac{h}{d}$, which we will hereafter denote by $\frac{1}{\epsilon}$, is the same as that of the resistances E A B K and E D K.

51. In all measurements of resistance it is necessary, not only that the contact of the electrodes A and B with the disk should be constant and easily reproduced, but also that the area of contact should be measurable; the difficulty of realizing these conditions with accuracy is by far the most important source of error to which the determinations of resistance are liable. In order to obtain a measurable and constant surface of contact, the spring contact-pieces employed in the experiments for plotting the equipotential lines were replaced by the following arrangement. A thin cylindrical rod of brass, to which two wires could be attached, slid vertically in a V-groove down the side of a rectangular wooden block. The lower end of the rod was filed accurately flat and perpendicular to its length, so as to touch the tinfoil with its entire surface. Constant pressure against the tinfoil was maintained by placing a weight on a cap fixed to its upper end. Two such rods constituted the electrodes A and B; and tolerably constant contacts could be made with them after a little practice. Nevertheless individual readings are apt to vary a good deal, and it is necessary to take the mean of several distinct experiments in order to get a single trustworthy result.

The numbers given in the annexed Table will serve to illustrate the variations occurring among single readings. They refer to a set of experiments in which the electrodes were placed approximately at the ends of a diameter of a circular disk.

	$2a$.	s.
	26·30	69·4
	26·22	69·7
	26·30	68·3
	26·30	68·3
	26·00	67·5
	26·31	68·7
	25·95	69·0
	26·25	68·6
	26·30	69·8
	26·00	69·4
	26·25	67·9
	26·31	68·7
	26·05	69·2
Means	.....26·20	68·80

52. The principal results of the measurements of resistance made by the method explained above are given in the Tables which follow, in such a way as to show how far they agree with the values indicated by theory. Every entry in the Tables is a mean value deduced from the number of experiments stated in the first column; the last column in each Table contains a quotient which theoretically should be constant. Common logarithms are employed, except when it is otherwise stated. The symbols used in the headings of the columns have the following significations:—

R, resistance in terms of an arbitrary unit (=·00155 ohm; being the resistance of one centimetre of the wire E F).

x, resistance of the contacts between the electrodes and the sheet (found by special experiments).

ϵ, ratio of the resistances EDF and EABF (figs. 1 or 3), or $\frac{d}{h}$ (§ 50).

s, shift of the slider D.

$2a$, or AB, distance between the poles.

ρ, radius of the circle of contact of the electrodes, or half the diameter of the brass rods.

r, radius of the tinfoil disk.

δ, thickness of the disk.

κ, specific conductivity of the material of disk.

L, the expression $\log_{10}\frac{AG \cdot BF \cdot A'G \cdot B'F}{\rho^2 \cdot A'F \cdot B'G}$ (see equation (20a), § 39, Part I.).

C, the constant $\frac{\log_e 10}{2\pi\kappa\delta}$.

The following relations hold good among the above quantities

$$R+x=\frac{s}{\epsilon}\ (\S\ 50)$$

and

$$R=CL\ (\S\S\ 38 \text{ and } 39, \text{ Part I.}),$$

whence

$$s-\epsilon x=\epsilon CL.$$

First set of Resistance experiments, April 1874.

ϵ was practically constant and $=5{\cdot}147$, $\rho={\cdot}163$, $\epsilon x=2{\cdot}42$.

Case 1. Two equal electrodes on the edge of a circular disk (Plate IV.).

Number of experiments.	r.	$2a$.	s.	$\frac{s-\epsilon x}{\log\frac{2a}{\rho}}$.
13	13·16	26·20	68·80	30·1
13		18·83	65 42	30·5
9		15·465	62·08	30·0
12		13·466	59·47	29·8
16		10·185	57·05	30·6
17	7·62	15·224	61·66	30·1
6		11·675	58·02	30·0
10		10·35	57·20	30·4
5		8·78	56·60	31·3
		Weighted mean.....		...30·29

In this case $L=4\log\frac{2a}{\rho}$ by formula (13), §§ 25 & 39, Part I., and therefore $\frac{s-\epsilon x}{\log\frac{2a}{\rho}}=4C\epsilon=$ const.: the mean value 30·29 gives $2C=2{\cdot}943$.

Case 2. Two equal electrodes, one at the centre and one on the edge of a circular disk.

Number of experiments.	r.	s.	$\frac{s-\epsilon x}{\log\frac{r}{\rho}}$
8	13·16	46·47	23·10
7	7·62	41·81	23·60
		Mean	.....23·33

Here $L=3\log\frac{r}{\rho}$ by formula (21), § 39, Part I., and therefore $\frac{s-\epsilon x}{\log\frac{r}{\rho}}=3C\epsilon=23{\cdot}33$, whence $2C=3{\cdot}022$.

The lines of flow and of equal potential for this case are shown in fig. 6, Plate V., by the part of the figure inside the circle. This figure represents the case of two opposite poles in an unlimited sheet, one of them being twice as strong as the other. One of the flow-lines is then a circle passing through the pole of double strength and having the other pole at its centre. The remaining flow-lines lie half of them within this circle and half of them outside. Consequently, if the sheet be cut along the circumference of this circle (which in the figure is shown as complete), a disk is obtained in which the electrical conditions are the same as in this experiment.

Case 3. Both electrodes in the disk, one on each side the centre and at equal distances from it.

Number of experiments.	r.	AB.	s.	$\frac{s-\epsilon x}{\log\frac{5\,.\,AB}{3\,.\,\rho}}$.
4	13·16	13·16	36·92	16·20
5	7·62	7·62	32·78	16·05
			Mean.....	16·12

Here L becomes $2\log\frac{5\,.\,AB}{3\,.\,\rho}$, whence $\frac{s-\epsilon x}{\log\frac{5\,.\,AB}{3\,.\,\rho}}=2C\epsilon$, and $2C=3{\cdot}132$.

Second set of Resistance experiments, April 1875.

Case 1. Both poles on the edge of the circular disk.

$r=13{\cdot}43$ centims., $\rho={\cdot}163$ centim., $x={\cdot}471$.

Number of experiments.	$2a$.	s.	ϵ.	$R=\frac{s}{\epsilon}-x$.	$\frac{R}{\log\frac{2a}{\rho}}$.
8	26·764	13·03	·9705	12·95	5·85
6	26·773	13·23	·9700	13·07	5·90
				Mean...	5·87

$L=4\log\frac{2a}{\rho}$; $\frac{R}{\log\frac{2a}{\rho}}=4C$; $2C=2{\cdot}935$.

Case 2. One pole at the centre, the other on the edge (as at *a*, fig. 4).

$r=13{\cdot}43$ centims., $\rho={\cdot}163$ centim., $x={\cdot}425$.

Number of experiments.	AB$=r$.	s.	ϵ.	$R=\frac{s}{\epsilon}-x$.	$\frac{R}{\log\frac{r}{\rho}}$.
3	13·43	9·37	1·035	8·63	4·504

$$L=3\log\frac{r}{\rho};\quad \frac{R}{\log\frac{r}{\rho}}=3C;\quad 2C=3{\cdot}003.$$

Case 3. Both poles inside the disk at distances CA, CB respectively from its centre.

$r=13{\cdot}43$ centims., $\rho={\cdot}163$ centim., $x={\cdot}38$.

AB.	CA.	CB.	s.	ϵ.	$R=\frac{s}{\epsilon}-x$.	$\frac{R}{L}$.
13·85	5·55	9·1	7·97	1·0595	7·14	1·63
20·00	10·9	9·1	8·94	1·0395	8·22	1·55
12·20	10·9	4·45	7·64	1·0625	6·81	1·58
7·25	4·5	3·62	6·45	1·0845	5·57	1·63
7·00	7·2	9·0	6·50	1·0850	5·62	1·61
9·00	0·0	9·0	7·17	1·0735	6·30	1·68
12·08	7·7	4·45	7·59	1·0650	6·75	1·64
15·12	7·7	7·7	8·04	1·0615	7·20	1·59
					Mean	...1·615

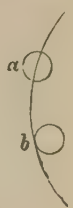
The flow-lines for this case are indicated in Plate V. fig. 9, where the circle with centre C corresponds with the edge of the disk. In order to get the theoretical resistance for this arrangement, additional poles, of the same signs as A and B respectively, must be assumed at the points A′ and B′; so that, if the conducting sheet were unlimited, the circle with centre C would coincide with a line of flow: that is to say, A′ and B′ must be on the prolongations of the radii through A and B respectively, and at the following distances from the centre, $CA'=\frac{r^2}{CA}$, $CB'=\frac{r^2}{CB}$ (see Part I. §§ 36 and 38). Taking

A and B as the middle points of the circles of contact made by the two electrodes, we have (*loc. cit.*)

$$L = \log \frac{(AB)^2 . AB' . A'B}{\rho^2 . AA' . BB'} ; \quad \frac{R}{L} = C ; \quad 2C = 3{\cdot}23.$$

The values of AB′ and A′B in the above expression were found by laying down on paper, to any convenient scale, the actual positions of the poles from the data AB, CA, CB, CA′, CB′, and then measuring the distances AB′ and A′B. This is a quicker process than the corresponding calculation.

Case 4. One electrode at the centre, the other *touching* the edge (as at *b*, fig. 4). Fig. 4.



$r = 13{\cdot}43$ centims., $\rho = {\cdot}163$ centim., $x = {\cdot}38$.

Number of experiments.	AB $= r - \rho$.	s.	ϵ.	$R = \frac{s}{\epsilon} - x$.	$\frac{R}{\log \frac{r-\rho}{\sqrt[3]{3} . \rho}}$.
7	13·27	9·13	1·038	8·41	4·80

The poles being assumed to be the centres of the circles of contact made by the electrodes,

$$L \text{ becomes } 3 \log \frac{r-\rho}{\sqrt[3]{3} . \rho} ; \quad \frac{R}{\log \frac{r-\rho}{\sqrt[3]{3} . \rho}} = 3C ; \quad 2C = 3{\cdot}20.$$

Case 5. Both electrodes *touching* the edge.

$r = 13{\cdot}43$ centims., $\rho = {\cdot}163$ centim., $x = {\cdot}38$.

Number of experiments.	AB.	s.	ϵ.	$R = \frac{s}{\epsilon} - x$.	$\frac{R}{\log \frac{AB}{\sqrt{3} . \rho}}$.
3	24·9	12·42	·983	12·25	6·30
1	16·8	11·58	·996	11·25	6·34
2	8·88	9·985	1·0065	9·54	6·36
2	10·68	10·59	1·016	10·043	6·36
2	22·0	12·34	·988	12·11	6·40
5	26·56	13·35	·969	13·40	6·79
				Mean ...	6·49

In this case, A and B being again taken as the middle points of the circular electrodes,

$$\text{L becomes } 4\log\frac{AB}{\sqrt{3}.\rho}; \quad \frac{R}{\log\frac{AB}{\sqrt{3}.\rho}}=4C; \quad 2C=3{\cdot}245.$$

53. *Summary of mean values of* $2C=\dfrac{\log_e 10}{\pi\kappa\delta}$.

Date	1874.		1875.		Mean.
	No. of experiments.	2C.	No. of experiments.	2C.	
Case 1. Both poles on the edge	101	2·943	14	2·935	2·94
Case 2. One in centre and one on the edge	15	3·02	3	3·00	3·01
Case 3. Both inside the disk	9	3·13	10	3·23	3·18
Case 4. One in centre and one touching the edge	.....	.....	7	3·20	3·20
Case 5. Both touching the edge	.....	.....	15	3·245	3·245

It will be seen by this Table that the values of C deduced from the several sets of experiments are not quite constant. In explanation of the discrepancies, it is to be observed that the equipotential lines determined by the contact of the electrodes with the tinfoil did not agree quite strictly with equipotential lines due to the arrangement of poles assumed in calculating the values of L, except in the experiments (distinguished as Case 1) where both the electrodes were on the edge of the disk. The differences between the results of these experiments and those with other arrangements of the electrodes are in the direction which would be caused by the actual departure from the theoretical conditions.

The discrepancy is greatest in Case 5, where both the electrodes *touch* the edge of the disk. In this case the additional poles A′ and B′, required in an imaginary unlimited continuation of the conducting sheet in order to make the circumference of the disk a line of flow, would be points outside the circumference, but very near to it and to A and B respectively. If, for

an instant, we regard the electrodes as mere points whose distance from the circumference of the disk is very small compared with its radius or with their distance from each other, the equipotential lines for a short distance from A and B will be very nearly indeed a system of lemniscates (see Part I. § 36, at the beginning)—the node of the self-cutting curve, or lemniscate proper, being sensibly on the circumference. The circle of contact formed by each electrode may be regarded as approximately representing one branch of such a lemniscate, if, instead of taking its centre for the actual pole (A or B), we take a point nearer the centre of the disk by $(\sqrt{2}-1)\rho$, where ρ is the radius of the electrode. A recalculation of the experiments of Case 5 in this way reduces the mean value of 2C from 3·245 to 3·11; and a similar correction applied to Case 4 gives 2C=3·10 instead of 3·20. Thus corrected, the results of these experiments agree fairly with those of Case 1.

54. Taking the mean value of 2C obtained with electrodes on the edge of the disk and multiplying it by $\frac{\pi}{\log_e 10}$, we get

4·011,

which ought to be the resistance of a square centimetre of the tinfoil used. For comparison with the value so found, the same quantity was determined by measuring the resistance of an oblong strip of tinfoil, the electrodes being flat pieces of brass, firmly clamped down on the ends of the strip, and making contact with its entire width. Five measurements, made with different lengths of a strip 3 centims. broad, gave as the resistance of a square centimetre

4·032,

the unit being, as before (see § 52), 0·00155 ohm; and five measurements, with a strip 1·5 centim. broad, gave

4·029.

The mean of these determinations is

4·031,

which agrees closely with the value calculated from the experiments with the disks.

55. It seems proper to point out in conclusion, that Professor J. Clerk Maxwell indicated so long ago as 1856 (Brit.-Assoc. Report, 1856, Trans. of Sect. p. 12; see also, for fuller developments, Camb. Phil. Trans. vol. x. p. 27) the fundamental idea of the method of treating questions relating to the flow of electricity in conductors of more than one dimension which was used in the first Part of this paper,—and also to mention that problems presented by finite plane conductors, in cases which become unmanageable by the simple geometrical processes employed by us (as, for instance, the case of rectangular conducting sheets), have been solved by Jochmann (Schlömilch's *Zeitschr. f. Mathemat.* 1865, p. 48) by a method founded on the superposition of the effects of separate poles, and by Heine (Crelle's *Journal*, 1874, vol. lxxix. p. 1) by an application of the geometrical principle of point-point correspondence (*Abbildung*) to Kirchhoff's results.

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1874–75.

March 21st, 1874.

Prof. GLADSTONE, F.R.S., President, in the Chair.

Capt. MARSHALL HALL, ALFRED B. HARDING, C. T. WOODWARD, B.Sc., H. W. EVE, C. B. BOURNE, M.A., J. RAE, M.D., LL.D., H. PERIGAL, F.R.A.S., LATIMER CLARK, C.E., F. CRISP, H. DEACON were admitted Members of the Society.

The following papers were read:—

"On the new Contact Theory of the Galvanic Cell." By J. A. FLEMING, B.Sc.

"On the Distribution of a Galvanic Current on entering and leaving a Conducting Medium." By Prof. F. GUTHRIE, F.R.S.

April 18th, 1874.

Prof. GLADSTONE, F.R.S., President, in the Chair.

J. H. S. GRAHAM, F.R.G.S., H. DURHAM, W. SPOTTISWOODE, F.R.S., Dr. H. MÜLLER, F.R.S., W. GARNETT, B.A., Rev. J. H. COOKE, F. CLOWES, B.Sc., E. CRAWSHAW, H. E. ARMSTRONG, PH.D., G. KING, C. L. FIELD, W. H. PIKE, Ph.D., A. S. HOBSON, S. LUPTON, B.A., A. S. L. MACDONALD, B.A., R. E. BAYNES, B.A., C. A. KESSELMEYER, W. J. KESSELMEYER, J. T. HURST, T. JONES, F.G.S., A. W. RÜCKER, M.A., were admitted Members of the Society.

The following papers were read:—

"On Wind Pressures in the Human Chest during performance on Wind Instruments." By Dr. W. H. Stone.

"On the Action of Hydrogen upon finely divided Metals." By A. Tribe, F.C.S.

Mr. F. Clowes, B.Sc., exhibited a simple form of Cell for the Lantern.

May 2nd, 1874.

Prof. Gladstone, F.R.S., President, in the Chair.

W. T. Goolden, B.A., T. Savage, M.A., Dr. Balfour Stewart, F.R.S., Prof. C. Lambert, M.A., T. Threlfall, M.A., A. J. Cownley were admitted Members of the Society.

(Council meeting only.)

May 9th, 1874.

Prof. Gladstone, F.R.S., President, in the Chair.

F. H. Varley, C.E., S. Forsyth, J. R. Adams were admitted Members of the Society.

The following papers were read:—

"On the Saltness or Freshness of the Ice formed on the Sea." By Dr. Rae.

"On the Transportation of Boulders from below Ice to its surface." By Dr. Rae.

"On the Fall in Pitch of Strained Wires through which a Galvanic Current is passing." By Dr. W. H. Stone.

"On a simple means of illustrating Wave Motion." By Mr. W. S. Davis.

May 23rd, 1874.

Prof. Gladstone, F.R.S., President, in the Chair.

F. E. Boughton, E. Rigg, B.A., Rev. A. Freeman, M.A., were admitted Members of the Society.

The following papers were read:—

"On a Modification of the Astatic Galvanometer." By Mr. Ridout.

"On an Absolute Galvanometer." By Prof. F. Guthrie, F.R.S.

June 13th, 1874.

Prof. GLADSTONE, F.R.S., President, in the Chair.

The following papers were read:—

"On an Apparatus for measuring Low Pressure of Gases." By Prof. M'LEOD.

"On an Arrangement by which the Coloured Rings of Uniaxal and Biaxal Crystals may be shown in a common Microscope fitted with the usual Nicol's Prisms." By Dr. W. H. STONE.

Discourse "On the Combination of Colours by Polarized Light." By W. SPOTTISWOODE, F.R.S.

June 20th, 1874.

Prof. GLADSTONE, F.R.S., President, in the Chair.

R. J. LECKY, F.R.A.S., BASIL WOODD Smith, F.R.A.S., J. STIRLING, M.A., H. TOMLINSON, B.A., R. WORMELL, M.A., D.Sc., A. C. RANYARD, C. W. SIEMENS, F.R.S., T. I. PEARSALL, S. E. PHILLIPS were admitted Members of the Society.

The following paper was read:—

"On Attraction and Repulsion accompanying Radiation." By W. CROOKES, F.R.S.

Mr. WOODWARD and Prof. BARRETT exhibited Instruments to illustrate the Interference of Sound.

November 7th, 1874.

Prof. W. G. ADAMS, F.R.S., Vice-President, in the Chair.

The following papers were read:—

"On an Instrument for Multiplying Small Motions." By G. F. RODWELL.

"On Graphical Methods of treating certain Elementary Electrical Problems." By Prof. G. C. FOSTER, F.R.S.

"On Salt Solutions and Water of Crystallization." By Prof. F. GUTHRIE, F.R.S.

November 21st, 1874.

Prof. GLADSTONE, F.R.S., President, in the Chair.

G. GRIFFITHS, M.A., W. HUGGINS, LL.D., F.R.S., R. W. OKES VOYSEY, F. D. BROWN, J. BAILLIE HAMILTON, R. MACCHEANE, G. E. WADE, and ALBERT MACKLIN were balloted for and elected Members of the Society.

The following papers were read:—

"On a Modified Voltaic Battery." By Prof. M'LEOD.

"On the application of Wind to Stringed Instruments." By J. BAILLIE HAMILTON.

December 5th, 1874.

Prof. GLADSTONE, F.R.S., President, in the Chair.

E. KINCH and J. F. WALKER, M.A., were balloted for and elected Members of the Society.

The following papers were read:—

"On a Strophometer." By T. HEARSON.

"On an Apparatus for determining the Coefficient of Expansion of a Solid Body." By Prof. G. C. FOSTER, F.R.S.

January 16th, 1875.

Prof. GLADSTONE, F.R.S., President, in the Chair.

W. BOTTOMLEY, W. H. PERKIN, F.R.S., and C. H. LEMANN were balloted for and elected Members of the Society.

The following papers were read:—

"On the Electrolysis of some Metallic Chlorides." By Prof. GLADSTONE, F.R.S., and A. TRIBE, F.C.S.

"On the Temperatures of Freezing Mixtures." By Prof. GUTHRIE, F.R.S.

January 30th, 1875.

Prof. G. C. FOSTER, F.R.S., Vice-President, in the Chair.

MOSTYN J. SALTER was balloted for and elected a Member of the Society

The following paper was read:—

"On Electrical Theories." By Dr. A. SCHUSTER.

Mr. C. BECKER exhibited a new Optical Bench.

Annual General Meeting, February 13th, 1875.

Prof. GLADSTONE, F.R.S., President, in the Chair.

The following Report of the Council was read by the President:—

For some years it has been a matter of regret among physicists that there was no Society specially devoted to the reading and discussion of papers connected with their science. The Royal Society readily receives important physical papers; but it is difficult to exhibit experiments, or to discuss them in detail, at Burlington House, and minor or unfinished papers are obviously unsuited for communication to this, the chief of the learned Societies.

A circular which Prof. GUTHRIE addressed in 1873 to the leading physicists elicited a gratifying number of replies, from which it was evident that many were ready to assist in the formation of such a Society, although there were various opinions as to the manner in which it should be constituted. It was, however, decided that it should not differ materially from other learned Societies; and on February 14th, 1874, a General Meeting accepted the Rules drawn up by the Provisional Committee, and requested its Members to act as the Officers and Council for the first year.

The Ordinary Meetings were commenced on the 21st of March under singularly favourable circumstances, as the Lords of the Committee of Council on Education generously placed at the disposal of the Society the Physical Lecture-room and Laboratories at South Kensington, as well as the valuable apparatus they contain. The Society was thus enabled to conduct the Meetings and to provide for the experimental illustration of papers at a very moderate cost, and to offer the advantages of Membership for the small Annual Subscription of £1. The number who have hitherto become Members is 138, of whom 28 have paid the Composition of £10, which entitles them to Life-Membership.

The Society has already to regret the loss of a gifted Member, Mr. W. S. DAVIS, of Derby, who died at the early age of thirty-two years. He was the Head-master of the Derby Central School of

Science; and during the last four years he taught with such skill and care that the study of physical science has been prosecuted with intelligent earnestness in the several classes he established. On the 9th of May he communicated a paper "On a simple means of illustrating wave-motion."

Most of the papers the titles of which are given above have been printed in our Proceedings; and in accordance with an arrangement made with the Editors of the 'Philosophical Magazine,' the original communications have the advantage of immediate publication in that journal.

The first part of the Proceedings was issued in October last, and two or three such parts may be expected annually.

The list shows that our papers have ranged over the sciences of Heat, Light, Sound, Electricity, Molecular and Mechanical Physics; and in the wide field of Physical Science new facts are constantly being discovered and new forms of apparatus devised. There is therefore good ground for hope that the Physical Society will always be abundantly furnished with papers.

The thanks of the Society are specially due to Mr. Spottiswoode, F.R.S., and to Mr. Baillie Hamilton, for discourses which they have delivered during the Session.

The desirability of undertaking the translation of valuable papers published abroad is under the consideration of the Council; and they are in communication with Prof. Everett, of Belfast, as to publishing an elaborate paper illustrating the "Centimetre-Gramme-Second" system of units.

The time of meeting has been carefully considered, as the Council were unwilling to add to the already numerous evening engagements of scientific men; and, on the other hand, were reluctant to encroach on the afternoon, which it is usual to keep as a holiday. Saturday afternoon has been chosen, but not without a fear that its selection may prove to be inconvenient to some Members. The Council feel, however, that on the whole the advantages of the time they have adopted outweigh the disadvantages; but they will in this, as in other things, hold themselves ready to carry out the clearly expressed wishes of the majority of the members.

Mr. Latimer Clark, in his recent inaugural address to the Society of Telegraphic Engineers, has referred to the suggestion which has been made that there should be an amalgamation of the two Societies, as it is probable that they will often receive papers of common interest. The great difficulty in carrying out the proposed

scheme lies in the fact that electricity is only one branch of physics, and that the practical details of telegraphy have little interest for the majority of our Members. On the other hand the Council are not unmindful of the advantages which would attend a close relation with so large and influential a body as the Telegraph Engineers.

The successful formation of our Society is only one indication of the attention which is now devoted to Physics; there is abundant evidence of the rapid rate at which its study is progressing in this country. The last Directory of the Science and Art Department shows that Electricity, Acoustics, Light, and Heat already attract nearly 20,000 students; and the number is increasing annually more rapidly than in the case of any other purely experimental scientific subject.

It is only necessary to add that our Society has been recognized by the Government in the formation of the Committee for carrying out the proposed Loan Exhibition of Scientific Apparatus at South Kensington.

The Society then proceeded to the election of Council and Officers for the ensuing year, and the following gentlemen were declared duly elected:

President.—Prof. J. H. GLADSTONE, F.R.S.

Vice-Presidents.—Prof. W. G. ADAMS, F.R.S.; Prof. G. C. FOSTER, F.R.S.

Secretaries.—Prof. A. W. REINOLD, M.A.; W. CHANDLER ROBERTS.

Treasurer.—Prof. E. ATKINSON.

Demonstrator.—Prof. F. GUTHRIE, F.R.S.

Other Members of Council.—LATIMER CLARK, C.E.; W. CROOKES, F.R.S.; Prof. A. DUPRÉ; Prof. O. HENRICI, F.R.S.; W. HUGGINS, F.R.S.; Prof. H. M'LEOD; W. SPOTTISWOODE, F.R.S.; Dr. H. SPRENGEL; Dr. W. STONE; E. O. WHITEHOUSE.

After the names of the Officers and Council had been announced from the Chair, votes of thanks were proposed to the Lords of the Committee of Council on Education for affording the Society the use of the rooms and apparatus at the Physical Laboratory, South Kensington, to the PRESIDENT, and to Prof. GUTHRIE for his valuable services to the Society.

The Treasurer in Account with the Physical Society, February 13th, 1875.

Dr.	£	*s.*	*d.*
122 Entrance-Fees in 1874	122	0	0
27 Life Compositions in 1874	270	0	0
95 Annual Subscriptions for 1874	95	0	0
2 Entrance-Fees in 1875	2	0	0
1 Life Composition for 1875	10	0	0
7 Subscriptions for 1875	7	0	0
	£506	0	0

Cr.	£	*s.*	*d.*	£	*s.*	*d.*
Stationery and Books:—						
Weaklin, Flint and Co.	3	9	3			
De La Rue and Co.	2	7	0			
Terry, Stoneman, and Co.	0	10	0			
Cash-Book	0	2	0			
Cheque-Book	0	5	0			
				6	13	3
Printing:—						
Taylor and Francis	28	15	6			
Richardson	3	1	6			
Taylor and Francis	4	3	7			
				36	0	7
Chapman, for attending Meetings in 1874				4	5	0
Postage, Receipt Stamps, Post-office Orders (1874):—						
Secretary	1	9	5			
Treasurer	1	10	6			
				2	19	11
Advertising				2	1	0
Balance in the Bank				454	0	3
				£506	0	0

Examined and found correct.

WILLIAM J. WILSON,
HERBERT M'LEOD, } *Auditors.*

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1875-76.

February 27th, 1875.

Dr. GLADSTONE, President, and afterwards Mr. LATIMER CLARK, in the Chair.

The following were elected Members of the Society:—

WILLIAM KEEP; C. G. MORGAN.

The following papers were read:—

"On a Method of Projecting the Sodium Spectrum on a Screen." By T. WILLS.

"On the Lines of Flow and Equipotential lines in a Uniform Conducting Sheet." By Prof. G. C. FOSTER and O. J. LODGE.

March 13th, 1875.

Dr. GLADSTONE, President, in the Chair.

The following were elected Members of the Society:—

Prof. J. D. EVERETT, D.C.L., F.R.S.E.; C. W. COOKE; T. A. HEARSON; Prof. A. B. W. KENNEDY.

The following papers were read:—

"On the Electro-Deposition of Iron." By W. C. ROBERTS.

"On Salt Solutions and Attached Water." By Prof. F. GUTHRIE.

April 10th, 1875.

Prof. G. C. Foster, Vice-President, in the Chair.

The following was elected a Member of the Society :—

Prof. W. Odling, M.A., F.R.S.

The following paper was read :—

"On the want of Achromatism of the Eye." By Prof. H. M'Leod.

Modified form of Kaleidoscope. By R. Cowper; exhibited by Prof. F. Guthrie.

Modification of Thomson's Quadrant Electrometer; exhibited by W. J. Wilson.

April 24th, 1875.

W. Spottiswoode, Esq., in the Chair.

The following papers were read :—

"On an Auxiliary Air-Pump." By Jerry Barrett.

"On some points connected with Wind Instruments." By Dr. W. H. Stone.

May 8th, 1875.

Dr. Gladstone, President, in the Chair.

The following paper was read :—

Exhibition of Experiments on Attraction and Repulsion accompanying Radiation. By W. Crookes.

"On the Determination of the Velocity of Light." By Prof. Cornu, of the École Polytechnique.

May 22nd, 1875.

Dr. Gladstone, President, in the Chair.

The following papers were read :—

"On a Revolving Polariscope." By W. Spottiswoode.

"On a new form of Polariscope." By Prof. W. G. Adams.

"On Fusion-points and Thermometry." By Dr. Mills.

"On the Electric Conductivity of Anthracite Coal." By H. Bauerman.

"On an Apparatus to illustrate the formation of Volcanic Cones." By Prof. Woodward.

June 12th, 1875.

Dr. GLADSTONE, President, in the Chair.

The following were elected Members of the Society:—

The Lord LINDSAY, M.P.; Prof. J. SYLVESTER, F.R.S.; Sir WILLIAM THOMSON, LL.D., D.C.L., F.R.S.

The following papers were read:—

"On the Electric Conductivity of Glass." By WILDMAN WHITEHOUSE.

"On the Time required for the Double Decomposition of Salts." By the PRESIDENT.

June 26th, 1875.

Prof. G. C. FOSTER, Vice-President, in the Chair.

The following were elected Members of the Society:—

The Earl of ROSSE, D.C.L., F.R.S.; LOUIS SCHWENDLER; R. S. BROUGH.

The following papers were read:—

"On a method of measuring the Electrical Resistance of Liquids." By W. J. WILSON.

"On the Subjective Phenomena of Taste." By Dr. W. H. STONE.

"On a new form of Micrometer for use in Spectrum Analysis." By Dr. W. M. WATTS.

"On the Fundamental Waves in Cylindrical Vessels." By Prof. F. GUTHRIE.

"On a new form of Magneto-Electric Machine." By S. C. TISLEY.

November 13th, 1875.

Dr. GLADSTONE, President, in the Chair.

The following paper was read:—

"On Thermopiles." By Dr. W. H. STONE.

November 27th, 1875.

Prof. G. C. FOSTER, Vice-President, in the Chair.

The following were elected Members of the Society:—

Prof. OSBORNE REYNOLDS, M.A.; Prof. H. J. S. SMITH, M.A., LL.D., F.R.S.; Prof. R. B. CLIFTON, M.A., F.R.S.; C. BUSK; J. M. THOMSON; J. W. W. WAGHORN; W. ESSON, M.A., F.R.S.; F. W. BAYLY; and Prof. R. W. EMERSON MACIVOR.

The following papers were read:—

"Description of Dr. Kerr's Researches on the Depolarizing Action of certain substances when under a powerful Electrical Tension." By Prof. Guthrie.

"On Stationary Liquid Waves." By Prof. Guthrie.

December 11th, 1875.

Dr. Gladstone, President, in the Chair.

The following were elected Members of the Society:—

C. Higgins and S. P. Thompson, B.A., B.Sc.

The following papers were read:—

"On a Simple Chronoscope for Measuring Short Intervals of Time." By O. J. Lodge for Prof. G. C. Foster.

"On an Arrangement for limiting the Charge given to a Leyden Jar." By Prof. M'Leod.

January 15th, 1876.

Dr. Gladstone, President, in the Chair.

The following were elected Members of the Society:—

Sir David Lionel Salomons, Bart.; Arthur R. Granville; and Capt. Abney, R.E.

The following papers were read:—

"On a new Apparatus for exhibiting Wave Motion." By Prof. Woodward.

"On some recent Methods of Spectroscopy." By J. N. Lockyer.

January 29th, 1876.

Dr. Gladstone, President, and afterwards Prof. G. C. Foster, Vice-President, in the Chair.

The following were elected Members of the Society:—

Sir John Conroy, Bart., M.A.; and H. T. Burls.

The following papers were read:—

"On the Polarization of Electrodes in Water free from Air." By J. A. Fleming.

"On the Photography of Fluorescent Substances." By the President.

"On Etheric Force." By S. P. Thompson.

Annual General Meeting, February 12th, 1876.

Dr. GLADSTONE, President, in the Chair.

The following were elected Members of the Society :—

W. R. HODGKINSON and H. M. HASTINGS.

The following Report of the Council was read by the President :—

At the last Annual General Meeting the Council was able to draw attention to the success which attended the formation of the Society, and to point to some of the ways for employing its resources with advantage in the future. They have now the satisfaction of stating that the activity with which the Society commenced its work has been well sustained during the past year, as is shown by the list of papers read, many of which have already appeared in the Proceedings.

With reference to communications which are more of the nature of lectures, the thanks of the Society are specially due to M. CORNU, of the École Polytechnique—who, on the 8th of May, described and illustrated the methods he had adopted for measuring the velocity of light,—and to Mr. J. NORMAN LOCKYER, for his discourse on some recent methods of Spectroscopy, which was delivered on the 15th of January last.

The number of Members has been increased from 138 to 169, of whom 35 have paid the Composition of £10, which entitles them to Life Membership.

The election of many distinguished Physicists during the past year has given the Council much satisfaction, as it affords undoubted evidence of the progress of the Society and of the position it has now attained.

The Society has to regret the loss of two Members, Mr. BECKER and Mr. WAUGH.

CARL LUDWIG CHRISTIAN BECKER was born at Ratzeburg, in the Grand Duchy of Mecklenburg-Strelitz, on the 16th of July, 1821. He received his general education at the Gymnasium of his birthplace, of which his father was Rector, and studied physical work with REPSOLD at Hamburg, KRAFT at Vienna, and STEINHEIL at Munich. He came to London in 1849, and joined the firm of ELLIOTT BROTHERS in 1858. He was a Fellow of the Royal Astronomical Society, a Member of the Society of Telegraph Engineers, and an Original Member of our Society. Mr. BECKER made a communication to the Society of Telegraph Engineers, in January 1875, on the Mance Heliograph; and he read a paper to this Society on a new form of

Optical Bench. He will best be remembered for the beauty and delicacy of the scientific appliances he designed, and for the untiring patience he displayed in constructing instruments to meet the special requirements of other physical workers. There are few of his contemporary Physicists who will not feel that by his death, which occurred on the 3rd of April, 1875, from bronchitis, they have lost a willing helper and a valued friend.

Alexander Waugh was grandson to the well-known Doctor of Divinity whose name he bore. He was born on the 29th of July, 1836, and educated under Dr. Payne, late President of the College of Preceptors. From an early age he showed great love for physical science, and also for music—two tastes not uncommonly united.

A course of lectures by Faraday in 1851 gave additional impulse to his studies in a scientific direction. He received from the great Physicist that kind attention and encouragement which were always granted by him to intelligent students. In 1855 Waugh entered at University College, and a few years later joined his father in an extensive Pharmaceutical business.

His favourite subject was Optics, especially spectroscopic research and the laws of Polarized Light. A paper by him "On the Spectrum of Polarized Light," was read before the Chemical Society, and also brought under the notice of the British Association by Dr. Gladstone. In this communication, as early as 1857, he described the beautiful absorption-bands to be seen traversing the spectrum of a polarized ray transmitted through a doubly refracting film. He constructed an apparatus for their demonstration and measurement, thus in a measure anticipating the later researches of Spottiswoode and Sorby. He also constructed with his own hands one of the finest spectroscopes yet made, in the manipulation of which he was singularly dexterous.

The sudden death by drowning of his only brother gave his health, always delicate, a severe shock; but, though ordered perfect repose, he was unable to tear himself from his favourite studies. On the 12th of October, 1875, an epileptiform seizure, indicating that the brain was physically unequal to its mental activity, terminated in death before his fortieth year. He was, however, no unworthy pupil of his great master Faraday.

In the last Report it was stated that the Council was in communication with Prof. Everett, of Belfast, as to the publication of an elaborate paper illustrating the Centimetre-Gramme-Second System of Units. This work, which was issued to Members in August last,

is based on the recommendations of a Committee of the British Association, and consists of a collection of physical data concisely presented on the above system, a complete account being added of the Theory of Units.

The Council is now in communication with the family of the late Sir CHARLES WHEATSTONE, with a view to the publication of his papers.

At the last General Meeting the number of Members of Council was increased from 13 to 17, and an additional Secretary was appointed—changes which have materially conduced to the efficient conduct of the Society's business.

During the last Session full abstracts of the papers and discussions have been furnished to the Scientific Journals by one of the Secretaries, who still continues to edit these reports, the preparation of which has been intrusted to a Member of the Society, whose services the Council has had the good fortune to secure.

A Committee of Council was appointed to revise the Bye-Laws, and it has proposed the changes which will be submitted to the Meeting today. This was considered necessary with a view to the formal registration of the Society in accordance with the provisions of the 23rd Section of the Companies' Act (1862) Amendment, 30 and 31 of Vic. Cap. 131, a—step which should not longer be delayed.

The Council is fully sensible of the advantage the Society derives from the use of the Physical Lecture-Room and Laboratories which have been so generously placed at its disposal by the Lords of the Committee of Council on Education, to whom the best thanks of the Members are due.

Friendly communications with other Physicists are of course most desirable; and the Society has already effected an interchange of publications with the Imperial University of Moscow and the Physical Societies of Paris and Berlin. The question of electing Foreign Members will receive due attention during the forthcoming Session.

The question of creating Associates of the Society has been discussed; but it was considered preferable that the Council should seek power to grant admission, to all Meetings of the Society, to persons who are not Members.

The Council desires, in concluding this Report, to record its gratitude for the important services which Dr. GUTHRIE has rendered to the Society. The Members will bear in mind that the office of Demonstrator was formerly an important one in the Royal Society; and the Council believes that much might be gained if arrangements

could be made for reproducing, before this Society, the experiments described in original papers which appear from time to time in this country and abroad.

Certain changes in the Bye-Laws were discussed and agreed to.

Messrs. A. S. Hobson and W. H. Walenn having been appointed Scrutators,

The Society proceeded to the election of the Council and Officers for the ensuing year; and the following gentlemen were declared duly elected:

President.—Prof. G. C. Foster, F.R.S.

Vice-Presidents.—Prof. W. G. Adams, F.R.S.; W. Spottiswoode, LL.D., F.R.S.

Secretaries.—Prof. A. W. Reinold, M.A.; W. C. Roberts, F.R.S.

Treasurer.—Prof. E. Atkinson.

Demonstrator.—Dr. F. Guthrie, F.R.S.

Other Members of Council.—Latimer Clark, C.E.; Prof. A. Dupré, F.R.S.; W. Huggins, D.C.L., F.R.S.; Prof. H. M'Leod; Dr. C. W. Siemens, D.C.L., F.R.S.; Dr. H. Sprengel; Dr. W. H. Stone; Sir William Thomson, LL.D., D.C.L., F.R.S.; Prof. W. C. Unwin, B.Sc.; E. O. W. Whitehouse.

After the names of the Council and Officers had been announced from the Chair, votes of thanks were passed:—to the Lords of the Committee of Council on Education for affording the Society the use of the rooms and apparatus at the Physical Laboratory, South Kensington; to the President; to Prof. Guthrie for his valuable services to the Society; and to the Treasurer and Secretaries.